

Chiral spin liquids and quantum error-correcting codes

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The possibility of using the twofold topological degeneracy of spin-1/2 chiral spin liquid states on the torus to construct quantum error-correcting codes is investigated. It is shown that codes constructed using these states on finite periodic lattices do not meet the necessary and sufficient conditions for correcting even a single qubit error with perfect fidelity. However, for large enough lattice sizes, these conditions are approximately satisfied, and the resulting codes may therefore be viewed as approximate quantum error-correcting codes.

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I. INTRODUCTION

If one could be built, a quantum computer would be capable of solving certain computational problems much more efficiently than any classical computer, most notably factoring large integers into primes in polynomial time [1] and searching unordered lists of N items in $O(N^{1/2})$ queries [2]. In a quantum computer classical bits, which take the values 0 or 1, are replaced by quantum bits, or qubits—two-level quantum systems whose Hilbert spaces are spanned by the orthonormal states $|0\rangle$ and $|1\rangle$. Unlike a classical bit, a qubit can therefore be placed in an arbitrary quantum superposition $\alpha|0\rangle + \beta|1\rangle$. However, due to the coupling of this qubit with the outside world, which may be small but which can never be reduced to zero, this state will eventually become entangled with its environment, losing its quantum coherence. Because maintaining this coherence is crucial for quantum computers to achieve their superiority over classical computers, the question of how to protect qubits from decoherence has been central to the ongoing effort in quantum computing.

One of the most surprising recent developments in quantum information theory has been the discovery of a scheme for fighting decoherence using what are called quantum error-correcting codes [3,4]. A quantum error-correcting code is a mapping from the Hilbert space of a single qubit (or, possibly, more than one qubit) to a subspace of the Hilbert space of many physical qubits. The resulting many-qubit state is then referred to as an *encoded* qubit. These encoded states are carefully designed so that if an error occurs, i.e., if a small number of the physical qubits become entangled with their environment, certain measurements can be performed to determine which error has occurred and how it can be corrected *without* disturbing the quantum information stored in the encoded qubit.

An important connection between quantum error-correcting codes, many-body physics, and topological quantum numbers was pointed out by Kitaev [5], who constructed a class of spin Hamiltonians realized on two-dimensional lattices with periodic boundary conditions. For the simplest of these models, the physical qubits correspond to spin-1/2 particles located on lattice edges. Kitaev showed that these models have degenerate ground states which are distinct, orthogonal states but which nevertheless cannot be distinguished by any *local* operators. Instead, these degenerate

states can only be distinguished by *global* operators, e.g., operators which act on a set of spins which encircle a topologically nontrivial cycle of the torus formed by the periodic lattice. Kitaev showed that these “topologically” degenerate states could be used as quantum error-correcting codes, called toric codes, in which encoded qubits correspond to particular superpositions of these degenerate ground states. Because these states can only be distinguished by topological quantum numbers, and not by any local observables, the environment has difficulty “measuring” the encoded quantum information, which is therefore, to some extent, protected from decoherence.

This connection between quantum spin models with topologically degenerate ground states and quantum error-correcting codes provides motivation to revisit some of the topological degeneracies which can arise “naturally” in certain condensed-matter systems. In this paper, as a concrete example, the spin-1/2 chiral spin liquid states originally proposed by Kalmeyer and Laughlin [6] as possible spin liquid ground states for frustrated spin-1/2 quantum antiferromagnets are considered in this context. It is known that these states possess a kind of topological order [7], not unlike the topological order of Kitaev’s toric codes, which leads to nontrivial ground-state degeneracies on Riemann surfaces with genus 1 or greater. It should be emphasized that it is by no means clear that the results of this paper will be useful for constructing quantum error-correction schemes for realistic quantum computers. Rather, the goal of the present work is to provide some insight into the possible ways that the Hilbert space of an array of qubits can exhibit topological quantum numbers.

The paper is organized as follows. In Sec. II, the basic physics of the chiral spin liquid states is reviewed. The case of finite $N_1 \times N_2$ periodic lattices is considered and it is proven that, if properly constructed, the chiral spin liquid states realized on these lattices are exact singlet states, generalizing a previous proof due to Laughlin that these states are singlets for $N \times N$ periodic lattices with N even [8]. In addition, it is shown by explicit construction that these states possess a topological degeneracy on any periodic lattice, in agreement with [7]. In Sec. III, the nature of this topological degeneracy is characterized using the Lieb-Schultz-Mattis “slow twist” operator and it is shown to be related to a topological decoupling of the Hilbert space of short-range valence-bond states on periodic lattices. In Sec. IV, the gen-

eral properties of quantum error-correcting codes are reviewed and compared with analogous properties of the topologically degenerate chiral spin liquid states, which are computed by variational Monte Carlo. It is shown that, unlike the toric codes, chiral spin liquid states do *not* satisfy the necessary and sufficient conditions to be exact quantum error-correcting codes. However, as the lattice size increases, the violation of these conditions becomes weaker until, in the thermodynamic limit, they become satisfied. In this sense the topologically degenerate chiral spin liquid states may be viewed as approximate quantum error-correcting codes on large enough lattices. Finally, Sec. V summarizes the results and conclusions of the paper.

II. CHIRAL SPIN LIQUID WAVE FUNCTIONS ON FINITE PERIODIC LATTICES

Consider the following spin-1/2 Hamiltonian realized on a two-dimensional square lattice:

$$H = J_1 \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + J_2 \sum_{\langle\langle i,j \rangle\rangle} \mathbf{S}_i \cdot \mathbf{S}_j, \quad (1)$$

where $J_1, J_2 \geq 0$ and $\langle i,j \rangle$ and $\langle\langle i,j \rangle\rangle$ denote nearest-neighbor and next-nearest-neighbor pairs of lattice sites, respectively. The lattice size is taken to be $N_1 \times N_2$ with N_1 even and lattice spacing b . Periodic boundary conditions will be assumed throughout the paper. For concreteness (and future reference) the lattice is taken to lie in the xy plane, with lattice sites $\mathbf{r} = (n_1 \hat{\mathbf{x}} + n_2 \hat{\mathbf{y}})b$, where n_1 and n_2 are integers.

If $J_1 > 0$ and $J_2 = 0$, Hamiltonian (1) describes an unfrustrated two-dimensional spin-1/2 Heisenberg antiferromagnet for which the ground state is known to possess long-range Néel order in the thermodynamic limit. In the opposite extreme, $J_2 \gg J_1$, the two sublattices decouple, and each develops Néel order independently. It is generally believed that over an intermediate range of J_2/J_1 values, the ground state is in a ‘‘spin-Peierls’’ phase with a locally observable broken translational symmetry, but there is no evidence that Hamiltonian (1) ever has a spin liquid ground state, i.e., a ground state with neither long-range Néel order nor any other locally observable broken translational symmetry, other than at zero-temperature critical points. Nevertheless, in what follows Hamiltonian (1) will be used to introduce the chiral spin liquid states with the understanding that while these states almost certainly do not describe the ground state of Eq. (1) in any parameter range, they *may* be eigenstates of an as yet unknown frustrated spin Hamiltonian. Fortunately, the topic of this paper—the relationship between chiral spin liquid states and quantum error-correcting codes—involves properties of Hilbert space and does not depend on the Hamiltonian.

Hamiltonian (1) can be viewed as describing a system of N interacting hard-core bosons hopping on a square lattice where $N = N_1 N_2 / 2$. In this description, the bosons correspond to up spins moving in a down spin background with matrix elements J_1 and J_2 for nearest-neighbor and next-nearest-neighbor hopping, respectively. If the totally sym-

metric wave function describing these bosons is $\Phi(\{\mathbf{r}_i\})$, then the corresponding spin state is

$$|\Phi\rangle = \sum_{\{\mathbf{r}_1 \mathbf{r}_2 \dots \mathbf{r}_N\}} \Phi(\{\mathbf{r}_i\}) S_{\mathbf{r}_1}^+ S_{\mathbf{r}_2}^+ \dots S_{\mathbf{r}_N}^+ |\downarrow \downarrow \dots \downarrow\rangle. \quad (2)$$

Because J_1 and J_2 are both positive, the effective hopping for the bosons is frustrated. As pointed out by Kalmeyer and Laughlin [6], this frustration can be viewed as being due to the presence of a magnetic field. To see this, imagine that each boson has charge q and moves in the presence of a magnetic field perpendicular to the plane of the lattice with field strength $B = \phi_0 / b^2$, where $\phi_0 = hc/q$ is the flux quantum. The corresponding vector potential in the Landau gauge is then $\mathbf{A} = -By \hat{\mathbf{x}}$ and it can readily be shown that

$$J_1 = J_1 \exp \frac{i2\pi}{\phi_0} \int_{\mathbf{r}_i}^{\mathbf{r}_j} \mathbf{A} \cdot d\mathbf{l}, \quad (3)$$

$$J_2 = -J_2 \exp \frac{i2\pi}{\phi_0} \int_{\mathbf{r}_i}^{\mathbf{r}_j} \mathbf{A} \cdot d\mathbf{l}, \quad (4)$$

where \mathbf{r}_i and \mathbf{r}_j denote the starting and ending sites of the relevant hopping process and the line integrals are taken along straight lines connecting these sites. Thus a positive (frustrated) J_2 corresponds to a negative (unfrustrated) J_2 in the presence of a fictitious magnetic field of suitable strength.

The sign of J_1 can be changed without affecting J_2 by dividing the square lattice into A and B sublattices and rotating the spins on the A sublattice by 2π radians about any fixed axis in spin space while leaving the B sublattice untouched. Under this sublattice rotation, the boson wave function is transformed according to

$$\Psi(\{\mathbf{r}_i\}) = \left(\prod_i e^{ib(x_i + y_i)/2} \right) \Phi(\{\mathbf{r}_i\}). \quad (5)$$

It is important for what follows to note that the spin wave function Φ must satisfy periodic boundary conditions on an $N_1 \times N_2$ lattice in the x and y directions, where N_1 is even. Therefore, for even values of N_2 the transformed wave function Ψ must also satisfy periodic boundary conditions in the x and y directions, while for odd values of N_2 , due to the sublattice mismatch, Ψ must satisfy periodic boundary conditions in the x directions and *antiperiodic* boundary conditions in the y direction.

This mapping from a frustrated spin system to a gas of hard-core bosons hopping on a lattice in the presence of a magnetic field inspired Kalmeyer and Laughlin to propose a trial wave function based on the related system of interacting bosons moving in free space [6]. Following their work, imagine that no lattice is present and that the bosons move on a torus of length $L_1 = N_1 b$ in the x direction and $L_2 = N_2 b$ in the y direction. Since the magnetic field is $B = \phi_0 / b^2 = 2(hc/q)n$, where n is the number density of bosons, the effective Landau-level filling fraction for the bosons is $\nu = 1/2$. In what follows, the effective magnetic

length $l_0 = (\hbar c/qB)^{1/2}$ is taken as the natural length scale and set equal to 1, so that, for example, the lattice spacing is $b = (2\pi)^{1/2}$.

A natural *ansatz* for the ground state of this many-boson system with strong short-range repulsion is that the bosons condense into a $\nu=1/2$ bosonic Laughlin state. The corresponding Laughlin wave function is completely determined by the lowest Landau-level constraint and the requirement that the wave function vanish as $\sim (z_i - z_j)^2$ as two bosons approach one another. In the Landau gauge, the Laughlin wave function for $\nu=1/2$ bosons on an $L_1 \times L_2$ torus can be written [9]

$$\Psi(\{\mathbf{r}_i\}) = \psi(\{z_i\}) \prod_i e^{-y_i^2/2}, \quad (6)$$

where

$$\psi(\{z_i\}) = F(Z) \prod_{i < j} \vartheta_1[\pi(z_i - z_j)/L_1 | \tau]^2. \quad (7)$$

Here $\vartheta_1(z|\tau)$ is the odd elliptic theta function [10] with $\tau = iL_2/L_1$, $z_i = x_i + iy_i$ is the complex coordinate of the i th boson, $Z = \sum_i z_i$ is the center-of-mass coordinate, and

$$F(Z) = e^{iKZ} \prod_{\nu=1}^2 \vartheta_1[\pi(Z - w_\nu)/L_1 | \tau] \quad (8)$$

is the center-of-mass part of the wave function.

The constants K , w_1 , and w_2 in Eq. (8) must be chosen so that Ψ satisfies the twisted boundary conditions

$$\Psi(\mathbf{r}_1 + \hat{\mathbf{x}}L_1, \mathbf{r}_2, \dots) = e^{i\phi_1} \Psi(\mathbf{r}_1, \mathbf{r}_2, \dots), \quad (9)$$

$$\Psi(\mathbf{r}_1 + \hat{\mathbf{y}}L_2, \mathbf{r}_2, \dots) = e^{i\phi_2} e^{-iL_2 x} \Psi(\mathbf{r}_1, \mathbf{r}_2, \dots), \quad (10)$$

for each boson coordinate. Here ϕ_1 and ϕ_2 are two toroidal fluxes which characterize the x and y boundary conditions. The requirement that the wave function satisfy these boundary conditions leads to the following restrictions on K , w_1 , and w_2 [9]:

$$e^{iKZ} = e^{i\phi_1}, \quad (11)$$

$$e^{i2\pi(w_1 + w_2)/L_1} = e^{i\phi_2} e^{kL_2}, \quad (12)$$

which can be satisfied in a variety of ways [9]. Here the $K = 0$ ‘‘coherent states’’ are used, for which

$$F_n(Z) = \vartheta_1[\pi(Z - W_n)/L_1 | \tau]^2, \quad (13)$$

where

$$W_n = \left(\frac{n}{2} + \frac{\phi_2}{4\pi} \right) L_1 \quad (14)$$

for $n=0$ and 1.

The two degenerate Laughlin states, Ψ_0 and Ψ_1 , corresponding to $n=0$ and 1, are distinguished only by the difference in the center-of-mass parts of their wave functions,

$F_n(Z)$. As shown by Haldane [11], this twofold degeneracy is required for any translationally invariant system on a torus at $\nu=1/2$. Note that although the states Ψ_0 and Ψ_1 span the two-dimensional Hilbert space of Laughlin states on the torus, they are not orthogonal. However, when $L_1 \gg L_2$ it can be shown that

$$F_n(Z) \approx \sqrt{\frac{L_2}{L_1}} \sum_{m=-\infty}^{\infty} \exp - \frac{\pi \left[Z - W_n - L_1 \left(m + \frac{1}{2} \right) \right]^2}{L_1 L_2}. \quad (15)$$

This function is sharply peaked at the points $Z = W_n + L_1/2 + mL_1$ for any integer m with peak widths $\sim \sqrt{L_1 L_2}$. It follows that in the limit $L_1/L_2 \rightarrow \infty$, the overlap between F_0 and F_1 vanishes and the states Ψ_0 and Ψ_1 become orthogonal.

To use Ψ_0 and Ψ_1 as wave functions for hard-core bosons on the $N_1 \times N_2$ periodic square lattice, the boson coordinates are restricted to the lattice points $\mathbf{r} = (n_1 \hat{\mathbf{x}} + n_2 \hat{\mathbf{y}})b$. For these lattice points, the $e^{-iL_2 x}$ factor in Eq. (10) is identically 1 and the toroidal fluxes ϕ_1 and ϕ_2 then correspond to overall phases associated with the boundary conditions in the x and y directions. As shown above, for even and odd values of N_2 the boson wave functions Ψ_n are required to be, respectively, periodic and antiperiodic, in the y direction. Thus $\phi_2 = 0$ for even values of N_2 and $\phi_2 = \pi$ for odd values of N_2 , implying that the parameter W_n appearing in $F_n(Z)$ is

$$W_n = \begin{cases} nL_1/2, & N_2 \text{ even,} \\ (2n+1)L_1/4, & N_2 \text{ odd.} \end{cases} \quad (16)$$

Because N_1 is assumed to be even, the wave functions Ψ_n are always required to be periodic in the x direction, implying that $\phi_1 = 0$ for all lattices.

The spin states Φ_n obtained from Ψ_n by undoing the sublattice rotation (5) are referred to as chiral spin liquid states. These states break both time-reversal symmetry (T) and parity (P) under both of which $\Phi \rightarrow \Phi^*$ [12]. This broken symmetry, characterized by a nonvanishing chiral order parameter $\langle \boldsymbol{\sigma}_1 \cdot (\boldsymbol{\sigma}_2 \times \boldsymbol{\sigma}_3) \rangle$, leads to a twofold degeneracy which is clearly *not* topological because it can be identified by measuring a local order parameter, and therefore not potentially useful for constructing quantum error-correcting codes. However, the additional degeneracy associated with the center-of-mass part of the wave function is global in character and not associated with any local order parameter (at least in the thermodynamic limit, see Sec. IV). Using a field-theoretic description, Wen has argued that the degeneracy of chiral spin liquids on a two-dimensional closed surface with genus g should be $2(2g)^8$, where the overall factor of 2 is due to the broken T and P symmetries and the factor of 2^8 is a measure of the topological order [7]. The explicit construction of the states Φ_0 and Φ_1 on periodic lattices is therefore consistent with Wen’s result for $g=1$.

The requirement that the spin states Φ_n be singlet states is equivalent to the requirement $S^-|\Phi_n\rangle=0$, which in turn implies that

$$\sum'_{\mathbf{r}_1} \Phi_n(\mathbf{r}_1, \mathbf{r}_2, \dots) = 0, \quad (17)$$

where the primed sum denotes a sum over lattice sites on the torus. If the toroidal fluxes are chosen so that Φ_n is periodic in both the x and y directions when evaluated on lattice sites, then it can be shown that Φ_n satisfies Eq. (17) by using the singlet sum rule derived by Laughlin [8],

$$\sum_{\mathbf{r}} G(\mathbf{r})f(z)e^{-|z|^2/2} = 0, \quad (18)$$

where for lattice sites $\mathbf{r} = (n_1\hat{\mathbf{x}} + n_2\hat{\mathbf{y}})b$,

$$G(\mathbf{r}) = (-1)^{n_1n_2+n_1+n_2+1}, \quad (19)$$

and $f(z)$ is any polynomial in z . Note that in order for Eq. (18) to be satisfied, it is necessary to sum over all lattice points on the infinite two-dimensional plane.

Following Laughlin [8], the sum rule (18) can be applied to the chiral spin liquid wave functions for finite $N_1 \times N_2$ periodic lattices by first exploiting the periodicity of Φ in the x and y directions to extend the summation in Eq. (17) to the entire lattice,

$$\sum'_{\mathbf{r}_1} \Phi_n(\mathbf{r}_1, \mathbf{r}_2, \dots) = \lim_{R \rightarrow \infty} \frac{2\pi N_1 N_2}{\pi R^2} \sum_{|\mathbf{r}_1| < R} \Phi_n(\mathbf{r}_1, \mathbf{r}_2, \dots), \quad (20)$$

and then using the following identity which holds for all lattice points:

$$e^{ib(x+y)/2} e^{-y^2/2} = -G(\mathbf{r}) e^{z^2/4} e^{-|z|^2/4}, \quad (21)$$

to show that

$$\begin{aligned} \sum_{|\mathbf{r}_1| < R} \Phi_n(\mathbf{r}_1, \mathbf{r}_2, \dots) &= \sum_{|\mathbf{r}_1| < R} G(\mathbf{r}_1) e^{z_1^2/4} \\ &\times F_n(Z) \psi(z_1, z_2, \dots) e^{-|z_1|^2/4} \\ &\times \prod_{i \neq 1} e^{-y_i^2/2}. \end{aligned} \quad (22)$$

In the limit $R \rightarrow \infty$, the summation on the right-hand side of Eq. (22) vanishes due to the sum rule (18) and the fact that the function $e^{z_1^2/4} F_n(Z) \psi(z_1, z_2, \dots)$ is analytic in z_1 . The chiral spin liquid states are therefore singlets for *any* $N_1 \times N_2$ lattice where N_1 is even, provided the toroidal fluxes have been chosen, as they have been here, to ensure that the spin wave function is periodic in both the x and y directions.

The nature of the topological degeneracy of the chiral spin liquid states depends on whether N_2 is even or odd. To understand this distinction, consider the translation operator T_x which translates each boson by one lattice vector in the x

direction. Under this operator, the relative part of the wave function Φ_n is unaffected and only the center-of-mass coordinate is shifted according to

$$Z \rightarrow Z + Nb = Z + \frac{N_1 N_2 b}{2} = Z + \frac{N_2 L_1}{2}. \quad (23)$$

For odd values of N_2 , this implies that the center of mass is shifted through a half-odd integer multiple of L_1 and, due to the periodic boundary conditions, this is equivalent to a net shift of the center of mass by $L_1/2$. Thus, in obvious notation,

$$T_x F_n(Z) = F_n(Z + L_1/2), \quad (24)$$

from which it follows that

$$T_x \Phi_0 = \Phi_1 \quad \text{and} \quad T_x \Phi_1 = \Phi_0. \quad (25)$$

This implies that for odd values of N_2 , the spin liquid states Φ_0 and Φ_1 break translational symmetry. In contrast, for even values of N_2 , the center-of-mass coordinate is shifted through an integer multiple of L_1 and the translation operator T_x has no effect,

$$T_x \Phi_0 = \Phi_0 \quad \text{and} \quad T_x \Phi_1 = \Phi_1. \quad (26)$$

Finally, because N_1 is even, for both even and odd values of N_2 ,

$$T_y \Phi_0 = \Phi_0 \quad \text{and} \quad T_y \Phi_1 = \Phi_1, \quad (27)$$

where T_y is the translation operator which translates each boson by one lattice vector in the y direction.

III. CONNECTION TO LIEB-SCHULTZ-MATTIS OPERATOR AND VALENCE-BOND TOPOLOGY

The topological degeneracy of the chiral spin liquid states can be elucidated further by introducing Affleck's two-dimensional generalization of the Lieb-Schultz-Mattis slow twist operator [13,14],

$$U_{\text{LSM}} = \exp\left(i \frac{\pi}{L_1} \sum'_{\mathbf{r}} x \sigma_{\mathbf{r}}^z\right), \quad (28)$$

where the primed sum denotes a sum over lattice points on the torus. The usefulness of this operator derives partly from the fact that when $N_1 \gg N_2$, for any singlet state $|\text{Sing}\rangle$ and any rotationally invariant spin Hamiltonian such as Eq. (1), which only includes short-range interactions, it can be shown that [13,14]

$$\langle \text{Sing} | (U_{\text{LSM}} H U_{\text{LSM}}^\dagger - H) | \text{Sing} \rangle \sim \mathcal{O}\left(J \frac{N_2}{N_1}\right), \quad (29)$$

where J is a measure of the typical magnetic interaction strength. If, as is supposed to be the case here, H has degenerate singlet ground states separated by a gap from all excited states, then Eq. (29) implies that in the $N_1/N_2 \rightarrow \infty$

limit, U_{LSM} maps states in the finite-dimensional Hilbert space spanned by these states into one another.

The Lieb-Schultz-Mattis slow twist operator can be recast in bosonic language as

$$U_{\text{LSM}} = \exp\left(-i\frac{\pi}{L_1} \sum_{\mathbf{r}}' x\right) \exp\left(i\frac{2\pi}{L_1} X\right), \quad (30)$$

where X is the x coordinate of the center of mass. Due to the periodic boundary conditions, there is some freedom in labeling the lattice sites on the torus, and in order to precisely define U_{LSM} it is necessary to choose a particular labeling scheme. Here it will be assumed that the primed sum in Eq. (30) is over lattice sites $\mathbf{r} = (n_1 \hat{\mathbf{x}} + n_2 \hat{\mathbf{y}})b$, where $n_1 = -N_1/2 + 1, \dots, N_1/2$ and $n_2 = 1, \dots, N_2$. For this choice, $\sum_{\mathbf{r}}' x = L_2 L_1 / (2b)$ and

$$U_{\text{LSM}} = (-i)^{N_2} \exp\left(i\frac{2\pi}{L_1} X\right). \quad (31)$$

As shown in Sec. II, when $L_1 \gg L_2$ the center-of-mass part of the chiral spin liquid wave functions Φ_n becomes sharply peaked for $Z = W_n + L_1/2 + mL_1$ for any integer m . Therefore, in this limit

$$\exp\left(i\frac{2\pi}{L_1} X\right) F_n(Z) \approx -\exp\left(i\frac{2\pi}{L_1} W_n\right) F_n(Z). \quad (32)$$

Combining Eqs. (31) and (32) and using Eq. (16), one finds that in the $N_1/N_2 \rightarrow \infty$ limit the states Φ_0 and Φ_1 become eigenstates of U_{LSM} with eigenvalues ± 1 ,

$$\lim_{(N_1/N_2) \rightarrow \infty} U_{\text{LSM}} \Phi_n = \begin{cases} (-1)^n (-1)^{(N_2+2)/2} \Phi_n, & N_2 \text{ even,} \\ (-1)^n (-1)^{(N_2+1)/2} \Phi_n, & N_2 \text{ odd.} \end{cases} \quad (33)$$

Figure 1 shows the results of a numerical variational Monte Carlo computation of the real part of the expectation values $\langle \Phi_n | U_{\text{LSM}} | \Phi_n \rangle$ for $n=0$ and 1 and $N_2=3$ and 4 plotted versus $1/N_1$. The results clearly show that $\text{Re}\langle \Phi_n | U_{\text{LSM}} | \Phi_n \rangle \rightarrow \pm 1$ in the $N_1 \rightarrow \infty$ limit for fixed N_2 . Because U_{LSM} is a unitary operator, it follows that Φ_0 and Φ_1 become eigenstates of U_{LSM} in the $N_1/N_2 \rightarrow \infty$ limit with eigenvalues ± 1 , consistent with Eq. (33).

The fact that Φ_0 and Φ_1 become eigenstates of U_{LSM} in the $N_1/N_2 \rightarrow \infty$ limit leads to an appealing picture of the topological degeneracy of chiral spin liquids in terms of the valence-bond state basis. This basis consists of states in which spins are singlet correlated in pairs, which are said to be connected by valence bonds. While *any* singlet state may be represented as a linear superposition of valence-bond states, it is reasonable to assume that any singlet state, such as the chiral spin liquid states, in which the spin-spin correlation function decays exponentially with distance [6,8], can be represented as a superposition of *short-range* valence-bond states. A short-range valence-bond state is a valence-bond state containing only bonds with lengths less than a specified length, or bonds with a distribution of lengths which falls off exponentially for long bonds.

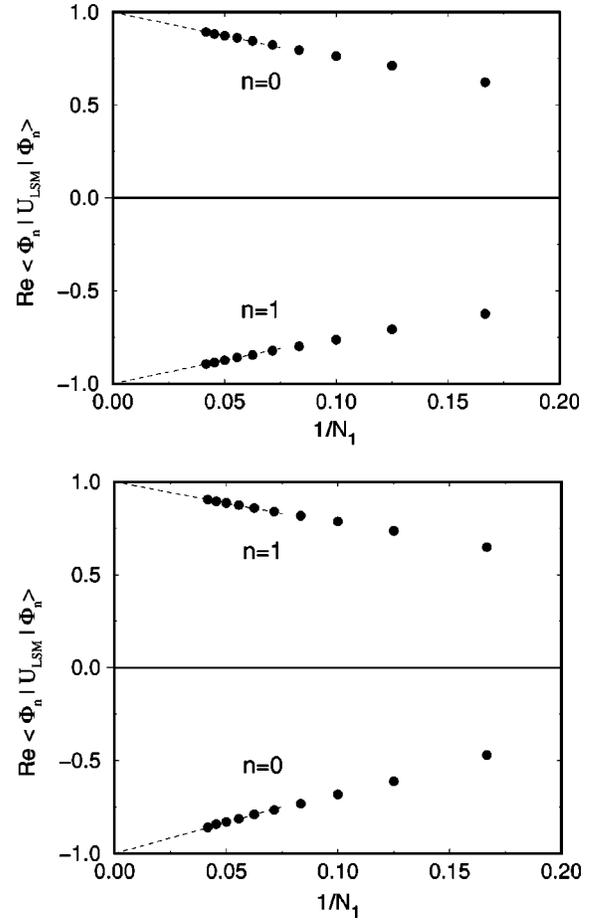


FIG. 1. Real part of the expectation value of the Lieb-Schultz-Mattis slow twist operator in the chiral spin liquid states Φ_0 and Φ_1 for $N_1 \times 3$ lattices (top) and $N_1 \times 4$ lattices (bottom) plotted vs $1/N_1$. Statistical error bars are smaller than symbol sizes.

The requirement that valence bonds must connect two sites, and only one bond may be attached to each site, gives rise to a topological decoupling of the space of short-range valence-bond states [15–18]. Figure 2 shows four short-range valence-bond states, two on a 6×3 lattice and two on a 6×4 lattice. In this figure solid lines connecting pairs of lattice sites represent valence bonds. In each of these states, the x projection of the length of each bond does not exceed $2b$ and so it is possible to unambiguously determine the way in which a given bond “wraps” around the periodic boundary condition in the x direction (it is in this sense that these states are short-range valence-bond states). For each of these states, six vertical dashed lines are shown which “slice” the gaps between each vertical line of lattice sites. The parity (o =odd, e =even) of the number of bonds cut by these dashed lines is then shown below each line.

For $N_2=3$, or any odd value of N_2 , an alternating even-odd pattern invariably appears [18]. Short-range valence-bond states then fall into two distinct classes, which can be referred to as even-odd and odd-even, corresponding to the two $N_2=3$ configurations shown in Fig. 2. For $N_2=4$, or any even value of N_2 , all gaps have the same parity and, again, there are two possibilities, either each gap has odd

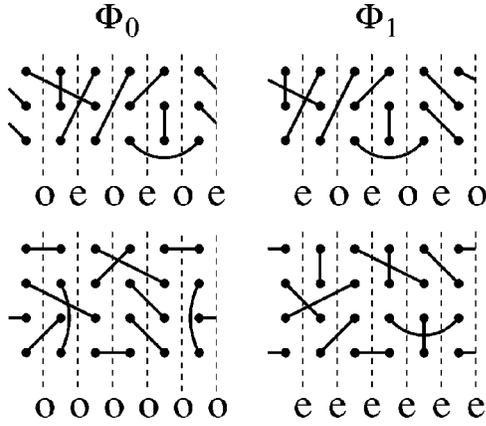


FIG. 2. Four short-range valence-bond configurations illustrating the topological quantum numbers responsible for the degeneracy of chiral spin liquid states on periodic lattices. Dashed lines are drawn through horizontal gaps in these configurations with the parity of the number of valence bonds crossed by each line shown below (e =even, o =odd). The upper two configurations on 6×3 lattices indicate the generic behavior for odd width lattices in which an alternating even-odd or odd-even pattern appears. The lower two configurations on 6×4 lattices indicate the generic behavior for even width lattices in which the gap parities are either all odd or all even. The configurations on the left and right contribute, respectively, to the states Φ_0 and Φ_1 , in a sense described in the text.

parity or each gap has even parity, corresponding to the two $N_2=4$ states shown in Fig. 2. It is convenient to define a topological quantum number, the gap parity, of a given short-range valence-bond state $|\alpha\rangle$ to be $(-1)^{\gamma_\alpha}$, where γ_α is the number of bonds which cross the gap between the line of lattice points with $x=L_1/2$ and those with $x=b-L_1/2$, i.e., those bonds which cross the discontinuity in x due to the periodic boundary conditions using the site-labeling scheme introduced above. According to this definition, for the two configurations at the top of Fig. 2, $\gamma_\alpha=2$ and 1, and the gap parities are $+1$ and -1 , while for the two configurations at the bottom of Fig. 2, $\gamma_\alpha=1$ and 2, and the gap parities are -1 and $+1$.

In [18] it was shown that if $N_1 \gg N_2$ for any short-range valence-bond state $|\alpha\rangle$,

$$U_{\text{LSM}}|\alpha\rangle \simeq (-1)^{\gamma_\alpha}|\alpha\rangle. \quad (34)$$

The appearance of the gap parity, $(-1)^{\gamma_\alpha}$, in Eq. (34) is due to the minus sign obtained whenever a spin-1/2 particle is rotated through 2π radians about any axis. If a given short-range valence-bond state is acted on by U_{LSM} , then, if $N_1 \gg N_2$, for most valence bonds in $|\alpha\rangle$ the two spins forming the bond are rotated by approximately the same amount. These valence bonds are therefore only weakly affected by the slow twist operator. However, for those γ_α bonds which cross the discontinuity in x due to the periodic boundary conditions, the operator U_{LSM} rotates one spin by approximately 2π radians while the other spin is, again approximately, not rotated at all. Therefore, while these bonds also

remain approximately in singlet states, they each contribute a factor of -1 to Eq. (34) because only *one* spin has been rotated through 2π radians.

According to Eq. (33), when $N_1 \gg N_2$ the states Φ_0 and Φ_1 become eigenstates of U_{LSM} with eigenvalues ± 1 . It is therefore plausible to assume that these states can be represented as linear superpositions of those short-range valence-bond states which also become eigenstates of U_{LSM} in this limit with the same eigenvalues, i.e., those states whose gap parities are equal to the corresponding eigenvalues given in Eq. (33). The gap parity can then be viewed as the topological quantum number which distinguishes between the states Φ_0 and Φ_1 in this limit. Note that the alternating even-odd or odd-even patterns which appear in the gap parities for odd values of N_2 , and the uniform gap parities, either all even or all odd, which appear for even values of N_2 , are consistent with the symmetry properties of Φ_0 and Φ_1 under the translation operators T_x derived in Sec. II. For more details on the connection between the Lieb-Schultz-Mattis slow twist operator and the topological decoupling of short-range valence-bond states, see [18].

IV. CONNECTION TO QUANTUM ERROR-CORRECTING CODES

In the preceding section it was shown that, in a sense which becomes precise in the limit $N_1 \gg N_2$, the topological quantum number distinguishing degenerate chiral spin liquid states is the gap parity. This topological quantum number is similar to that of Kitaev's toric code in that it appears to be necessary to measure a *global* property of the system, using, for example, the Lieb-Schultz-Mattis slow twist operator, in order to determine its value. Motivated by this similarity between chiral spin liquids and toric codes, it is natural to ask whether, or to what extent, the topologically degenerate chiral spin liquid states on finite lattices can be viewed as quantum error-correcting codes.

A quantum error-correcting code for a single qubit is a mapping of the form, $|0\rangle \rightarrow |0_L\rangle$ and $|1\rangle \rightarrow |1_L\rangle$, where the states $|0_L\rangle$ and $|1_L\rangle$ are made up of several *physical* qubits. If the encoded qubit is placed in a pure state $|\Upsilon_L\rangle = \alpha|0_L\rangle + \beta|1_L\rangle$, then the initial density matrix describing the state is $\rho_0 = |\Upsilon_L\rangle\langle\Upsilon_L|$. After the physical qubits making up the encoded qubit interact with their environment, the most general effect on the density matrix is

$$\rho_0 \rightarrow \sum_a E_a \rho_0 E_a^\dagger = \rho_E = \sum_a E_a |\Upsilon_L\rangle\langle\Upsilon_L| E_a^\dagger, \quad (35)$$

with the constraint $\sum_a E_a^\dagger E_a = 1$, where the operators E_a are referred to as error operators. In order to be able to return the encoded qubit to its original pure state, there must exist a recovery operation which satisfies

$$\rho_E \rightarrow \sum_a R_a \rho_E R_a^\dagger = \rho_0 = |\Upsilon_L\rangle\langle\Upsilon_L|, \quad (36)$$

TABLE I. Nearest-neighbor spin-spin correlation functions in the states Φ_0 and Φ_1 for different lattice sizes.

Lattice size $N_1 \times N_2$	$\langle \Phi_0 \sigma_{\mathbf{r}_0}^z \sigma_{\mathbf{r}_0 + b\hat{x}}^z \Phi_0 \rangle$	$\langle \Phi_0 \sigma_{\mathbf{r}_0}^z \sigma_{\mathbf{r}_0 + b\hat{y}}^z \Phi_0 \rangle$	$\langle \Phi_1 \sigma_{\mathbf{r}_0}^z \sigma_{\mathbf{r}_0 + b\hat{x}}^z \Phi_1 \rangle$	$\langle \Phi_1 \sigma_{\mathbf{r}_0}^z \sigma_{\mathbf{r}_0 + b\hat{y}}^z \Phi_1 \rangle$
4 × 2	-0.173(2)	-0.946(2)	-0.455(2)	0.273(5)
4 × 4	-0.247(2)	-0.246(3)	-0.230(2)	-0.376(3)
4 × 6	-0.216(2)	-0.312(2)	-0.217(2)	-0.301(3)
4 × 8	-0.210(2)	-0.306(3)	-0.210(2)	-0.307(3)
6 × 2	-0.176(2)	-0.944(2)	-0.467(2)	0.322(5)
6 × 4	-0.311(2)	-0.216(3)	-0.279(2)	-0.376(3)
6 × 6	-0.298(2)	-0.300(3)	-0.302(2)	-0.281(3)
6 × 8	-0.303(2)	-0.289(3)	-0.302(2)	-0.290(3)
8 × 2	-0.175(2)	-0.944(2)	-0.464(2)	0.335(5)
8 × 4	-0.306(2)	-0.210(3)	-0.275(2)	-0.382(3)
8 × 6	-0.290(2)	-0.303(3)	-0.292(2)	-0.281(3)
8 × 8	-0.291(2)	-0.291(2)	-0.290(2)	-0.293(3)
4 × 3	-0.230(2)	-0.241(3)	-0.301(2)	-0.241(3)
4 × 5	-0.229(2)	-0.301(3)	-0.221(2)	-0.301(3)
4 × 7	-0.213(2)	-0.305(3)	-0.213(2)	-0.305(3)
4 × 9	-0.209(2)	-0.306(3)	-0.209(2)	-0.306(3)
6 × 3	-0.334(2)	-0.239(3)	-0.257(2)	-0.239(3)
6 × 5	-0.280(2)	-0.290(3)	-0.292(2)	-0.290(3)
6 × 7	-0.283(2)	-0.294(3)	-0.281(2)	-0.294(3)
6 × 9	-0.281(2)	-0.293(3)	-0.281(2)	-0.293(3)
8 × 3	-0.258(2)	-0.239(3)	-0.336(2)	-0.239(3)
8 × 5	-0.298(2)	-0.290(3)	-0.293(2)	-0.290(3)
8 × 7	-0.291(2)	-0.290(3)	-0.292(2)	-0.290(3)
8 × 9	-0.290(2)	-0.291(3)	-0.290(2)	-0.291(3)

again with the constraint $\sum_a R_a^\dagger R_a = 1$. The necessary and sufficient conditions for such a recovery operation to exist are [19,20]

$$\langle 0_L | A_a^\dagger A_b | 0_L \rangle = \langle 1_L | A_a^\dagger A_b | 1_L \rangle, \quad (37)$$

$$\langle 0_L | A_a^\dagger A_b | 1_L \rangle = 0, \quad (38)$$

where the set of operators $\{A_a\}$ form a linear basis for the error operators, i.e., every error operator can be expanded as $E_a = \sum_b \lambda_{ab} A_b$. For example, for a code capable of correcting only a single qubit error, one may take the basis $\{A_a\}$ to consist of the identity operator and all Pauli matrices acting on individual physical qubits.

The chiral spin liquid states Φ_0 and Φ_1 are not orthogonal, except in the limit $N_1/N_2 \rightarrow \infty$. However, on any finite lattice it is possible to orthogonalize them and use the resulting states as a quantum code where the spin-1/2 particles located at lattice sites correspond to the physical qubits. For example,

$$|0_L\rangle = |\Phi_0\rangle, \quad (39)$$

$$|1_L\rangle = \frac{|\Phi_1\rangle - \langle \Phi_0 | \Phi_1 \rangle |\Phi_0\rangle}{(1 - |\langle \Phi_0 | \Phi_1 \rangle|^2)^{1/2}}. \quad (40)$$

The question to be addressed is then, to what extent do these states satisfy the criteria (37) and (38) for being quantum error-correcting codes?

Because Φ_0 and Φ_1 are singlets, it is possible to simplify Eqs. (37) and (38) considerably for the case of single qubit errors by noting that an arbitrary encoded qubit $|Y_L\rangle = \alpha|0_L\rangle + \beta|1_L\rangle$ is also a singlet, implying that

$$\langle Y_L | \sigma_{\mathbf{r}_i}^\alpha | Y_L \rangle = 0 \quad (41)$$

for all lattice sites \mathbf{r}_i , where $\alpha = x, y, \text{ or } z$, thus ensuring that Eqs. (37) and (38) are satisfied for $A_a = \sigma_{\mathbf{r}_i}^\alpha$ and $A_b = 1$. Likewise,

$$\langle Y_L | \sigma_{\mathbf{r}_i}^\alpha \sigma_{\mathbf{r}_j}^\beta | Y_L \rangle = \delta_{\alpha\beta} \langle Y_L | \sigma_{\mathbf{r}_i}^z \sigma_{\mathbf{r}_j}^z | Y_L \rangle \quad (42)$$

for all lattice sites \mathbf{r}_i and \mathbf{r}_j . The conditions for a singlet state to be a quantum error-correcting code capable of correcting a single qubit error can then be shown to be equivalent to the requirement that

$$\langle Y_L | \sigma_{\mathbf{r}_i}^z \sigma_{\mathbf{r}_j}^z | Y_L \rangle = C_{ij} \quad (43)$$

for *all* states Y_L , i.e., the spin-spin correlation functions must be *identical* for any encoded state.

Because the spin-spin correlation function decays rapidly with distance in the chiral spin liquid states [6,8], the largest violation of Eq. (43) is likely to occur for nearest-neighbor spin correlations. Consider these correlations for $|Y_L\rangle = |\Phi_0\rangle$ and $|Y_L\rangle = |\Phi_1\rangle$. For odd values of N_2 , there is a broken translation symmetry in the x direction, $T_x \Phi_0 = \Phi_1$ and $T_x \Phi_1 = \Phi_0$, and, for any lattice site \mathbf{r}_0 ,

$$\langle \Phi_n | \sigma_{\mathbf{r}_0}^z \sigma_{\mathbf{r}_0 + b\hat{\mathbf{a}}}^z | \Phi_n \rangle = \langle \Phi_n | \sigma_{\mathbf{r}_0 + 2b\hat{\mathbf{x}}}^z \sigma_{\mathbf{r}_0 + b\hat{\mathbf{a}} + 2b\hat{\mathbf{x}}}^z | \Phi_n \rangle \quad (44)$$

and

$$\langle \Phi_0 | \sigma_{\mathbf{r}_0}^z \sigma_{\mathbf{r}_0 + b\hat{\mathbf{a}}}^z | \Phi_0 \rangle = \langle \Phi_1 | \sigma_{\mathbf{r}_0 + b\hat{\mathbf{x}}}^z \sigma_{\mathbf{r}_0 + b\hat{\mathbf{a}} + b\hat{\mathbf{x}}}^z | \Phi_1 \rangle. \quad (45)$$

Here, and in what follows, $\hat{\mathbf{a}} = \hat{\mathbf{x}} \cdot \hat{\mathbf{y}}$. For even values of N_2 , there is no broken translation symmetry in the x direction, $T_x \Phi_n = \Phi_n$, and

$$\langle \Phi_n | \sigma_{\mathbf{r}_0}^z \sigma_{\mathbf{r}_0 + b\hat{\mathbf{a}}}^z | \Phi_n \rangle = \langle \Phi_n | \sigma_{\mathbf{r}_0 + b\hat{\mathbf{x}}}^z \sigma_{\mathbf{r}_0 + b\hat{\mathbf{a}} + b\hat{\mathbf{x}}}^z | \Phi_n \rangle. \quad (46)$$

For both even and odd values of N_2 , there is no broken translation symmetry in the y direction, $T_y \Phi_n = \Phi_n$, and so in both cases

$$\langle \Phi_n | \sigma_{\mathbf{r}_0}^z \sigma_{\mathbf{r}_0 + b\hat{\mathbf{a}}}^z | \Phi_n \rangle = \langle \Phi_n | \sigma_{\mathbf{r}_0 + b\hat{\mathbf{y}}}^z \sigma_{\mathbf{r}_0 + b\hat{\mathbf{a}} + b\hat{\mathbf{y}}}^z | \Phi_n \rangle. \quad (47)$$

Finally, for odd values of N_2 the fact that the chiral spin liquid states are symmetric under PT , the product of parity and time-reversal [12], implies that

$$\langle \Phi_0 | \sigma_{\mathbf{r}_0}^z \sigma_{\mathbf{r}_0 + b\hat{\mathbf{y}}}^z | \Phi_0 \rangle = \langle \Phi_1 | \sigma_{\mathbf{r}_0}^z \sigma_{\mathbf{r}_0 + b\hat{\mathbf{y}}}^z | \Phi_1 \rangle. \quad (48)$$

Though it is not possible to compute these correlation functions analytically, it is straightforward to compute them numerically using standard variational Monte Carlo techniques, and the results of such calculations for various lattice sizes are given in Table I. In this table the site \mathbf{r}_0 is taken to be the origin ($\mathbf{r}_0 = 0\hat{\mathbf{x}} + 0\hat{\mathbf{y}}$) and for each lattice size the correlation functions $\langle \Phi_n | \sigma_{\mathbf{r}_0}^z \sigma_{\mathbf{r}_0 + b\hat{\mathbf{x}}}^z | \Phi_n \rangle$ and $\langle \Phi_n | \sigma_{\mathbf{r}_0}^z \sigma_{\mathbf{r}_0 + b\hat{\mathbf{y}}}^z | \Phi_n \rangle$ are given for $n=0$ and 1. Using the symmetry properties derived above, these correlation functions can be used to determine all nearest-neighbor spin-spin correlation functions for Φ_0 and Φ_1 . (Note that for odd values of N_2 there is some redundancy in the table, since $\langle \Phi_0 | \sigma_{\mathbf{r}_0}^z \sigma_{\mathbf{r}_0 + b\hat{\mathbf{y}}}^z | \Phi_0 \rangle = \langle \Phi_1 | \sigma_{\mathbf{r}_0}^z \sigma_{\mathbf{r}_0 + b\hat{\mathbf{y}}}^z | \Phi_1 \rangle$.)

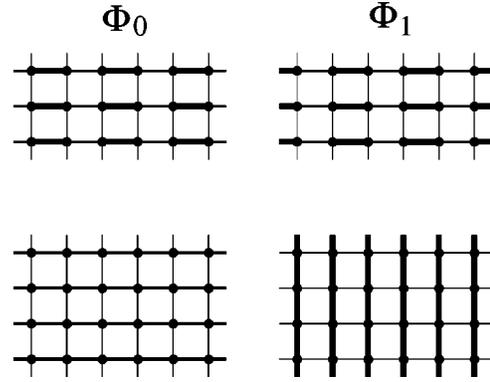


FIG. 3. Patterns formed by nearest-neighbor spin-spin correlation functions $\langle \Phi_n | \sigma_{\mathbf{r}_i}^z \sigma_{\mathbf{r}_j}^z | \Phi_n \rangle$ for the topologically degenerate chiral spin liquid states Φ_0 and Φ_1 on 6×3 and 6×4 lattices. Thicker lines correspond, qualitatively, to larger values of $-\langle \Phi_n | \sigma_{\mathbf{r}_i}^z \sigma_{\mathbf{r}_j}^z | \Phi_n \rangle$ (thicknesses of the lines are exaggerated for clarity). For odd values of N_2 , the broken translation symmetry is observable. For even values of N_2 , the nearest-neighbor spin-spin correlation functions are different in the two states. The ability to distinguish between Φ_0 and Φ_1 by measuring operators consisting of only two Pauli matrices indicates that although the underlying distinction between them is topological, as depicted in Fig. 2, these states are not quantum error-correcting codes on finite lattices.

As can be seen in Table I, on finite lattices the nearest-neighbor spin-spin correlation functions are *not* identical for Φ_0 and Φ_1 , thus violating Eq. (43). Therefore, on these lattices, the topologically degenerate chiral spin liquid states are not exact quantum error-correcting codes, even for single qubit errors. While it is true that with increasing lattice size the difference between correlation functions in Φ_0 and Φ_1 becomes smaller, until it is no longer possible to distinguish between them due to the statistical error bars of the Monte Carlo simulation, given the clear violation of Eq. (43) for lattices sizes as large as 8×6 it is unlikely that these correlation functions ever become exactly equal to one another on any finite lattice. Rather, it is more plausible that they approach each other exponentially as the system size, in particular N_2 , increases, though no proof of this has been found.

The distinction between the states Φ_0 and Φ_1 can be seen clearly in the patterns formed by the values of the nearest-neighbor spin-spin correlation functions. These patterns are shown for 6×3 and 6×4 lattices in Fig. 3. If the topologically degenerate chiral spin liquid states did provide exact quantum error-correcting codes for single qubit errors, then these patterns would be identical for a given lattice size. Figure 3 together with Table I show clearly that despite the fact that the underlying distinction between the states Φ_0 and Φ_1 is topological, as illustrated in Fig. 2, on finite lattices the difference between them can still be measured locally using just two Pauli matrices. However, as stated above, the nearest-neighbor spin-spin correlation functions rapidly become effectively indistinguishable for these two states as the lattice size increases, as do, plausibly, all the correlation functions appearing in Eq. (43). In this sense the topologically degenerate chiral spin liquid states on sufficiently large lattices may be viewed as approximate quantum error-correcting codes.

V. CONCLUSIONS

In this paper, the chiral spin liquid states first introduced by Kalmeyer and Laughlin as possible ground states for frustrated spin-1/2 antiferromagnets have been analyzed from the point of view of their connection to quantum error-correcting codes. Explicit wave functions were constructed for the two topologically degenerate chiral spin liquid states on finite periodic $N_1 \times N_2$ lattices with N_1 even and it was proven that, if properly constructed, these states are exact singlets for any such lattice. It was also shown that, in a sense which becomes precise when $N_1 \gg N_2$, the property characterizing the topological degeneracy is the gap parity—a topological quantum number associated with the short-range valence-bond state basis. However, despite the fact that, like Kitaev's toric codes, the degenerate chiral spin liquid states are distinguished by a topological quantum number, these states are

not perfectly indistinguishable when measured with local operators, except in the thermodynamic limit. Thus, on finite periodic lattices, these states do not satisfy the criteria (37) and (38), and so are not exact quantum error-correcting codes—any error-correction scheme using chiral spin liquid states would not be able to recover even a single qubit error with perfect fidelity. At the same time, the distinction between these states, as measured by local operators, rapidly becomes unobservable as the lattice size increases. Therefore, on large enough lattices, the topologically degenerate chiral spin liquid states may be viewed as approximate quantum error-correcting codes.

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