Spin correlations and topological entanglement entropy in a non-Abelian spin-one spin liquid

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We analyze the properties of a non-Abelian spin-one chiral spin liquid state proposed by Greiter and Thomale [Phys. Rev. Lett. **102**, 207203 (2009)] using Monte Carlo. In this state the bosonic $\nu = 1$ Moore-Read Pfaffian wave function is used to describe a gas of bosonic spin flips on a square lattice with one flux quantum per plaquette. For toroidal geometries there is a three-dimensional space of these states corresponding to the topological degeneracy of the bosonic Moore-Read state on the torus. We show that spin correlations for different states in this space become indistinguishable for large system size. We also calculate the Renyi entanglement entropy for different system partitions to extract the topological entanglement entropy and provide evidence that the topological order of the lattice spin-liquid state is the same as that of the continuum Moore-Read state from which it is constructed.

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Introduction. Fractional quantum Hall states are prototypical examples of topologically ordered states of matter [1]—states which are not characterized by local order parameters, but rather by ground state degeneracies on topologically nontrivial surfaces and fractionalized excitations. These excitations are predicted to be anyons, in most case obeying (fractional) Abelian statistics, as in the original Laughlin state, but also, possibly, non-Abelian statistics, as in the Moore-Read Pfaffian state [2]. Recent work on the non-Abelian case has been driven not only by its intrinsic interest but also by the possibility that the resulting non-Abelian anyons could be used for topological quantum computation [3,4]. This has motivated the search for possible realizations of states with non-Abelian topological order beyond fractional quantum Hall states, with one promising class of such states being the quantum spin liquids.

The notion of quantum spin liquids, possible ground states of frustrated quantum antiferromagnets with no conventional long-range magnetic order, can be traced back to the original triangular lattice RVB state proposed by Anderson [5]. Examples of theoretically established Abelian spin liquids which are total spin singlets and have been shown to be ground states of explicit local Hamiltonians include an $SU(2)$-invariant $\mathbb{Z}_2$ quantum spin liquid on the kagome lattice [6–10] and the Abelian chiral spin liquid (CSL) introduced by Kalmeyer and Laughlin [11,12] for which Hamiltonians have been constructed in Refs. [13–15].

The Abelian CSL state is a spin-1/2 spin liquid that can be constructed using a continuum Laughlin wave function for bosons to describe the amplitudes for spin flips on a lattice. In this paper we investigate properties of a possible spin-one CSL state proposed by Greiter and Thomale [16] that is similarly based on the continuum Moore-Read wave function [2] known to have non-Abelian topological order with Ising anyon excitations. Model Hamiltonians for which this state becomes a ground state in the thermodynamic limit have been constructed [15,17]. However, to firmly establish that the state itself is, indeed, a spin liquid with non-Abelian topological order it is necessary to show that (i) it has exponentially decaying spin correlations, (ii) it has the expected topological degeneracy, and (iii) the restriction of bosons to the lattice does not destroy the non-Abelian topological order of the continuum state. Entanglement properties on the cylinder studied in Ref. [17] have suggested that both the Abelian and non-Abelian CSL states harbor the same topological order as their continuum parents. Here, we study the non-Abelian CSL for both planar and toroidal geometries (the latter being necessary to study topological degeneracy and related modular properties) and provide compelling evidence that it is indeed a quantum spin liquid with exponentially decaying spin correlations and the same modular $\mathcal{S}$ matrix (and hence the same non-Abelian topological order) as the continuum Moore-Read state.

Non-Abelian CSL state on planar geometry/torus. We begin by reviewing the spin-one non-Abelian CSL state for planar geometry proposed by Greiter and Thomale [16]. This state is constructed using the bosonic Moore-Read state [2] with filling fraction $\nu = 1$ for which the droplet wave function in the symmetric gauge is

$$\Psi[z_i] = \mathrm{Pf}\left(\frac{1}{z_j - z_k}\right)\prod_{i<j}^{N}(z_i - z_j)\prod_{i}^{N} e^{-|z_i|^2/4}. \qquad (1)$$

We work in units with magnetic length equal to 1 for which the square lattice formed by points with complex coordinates $z = \eta_{nm} = \sqrt{2\pi}(n + im)$ where n and m are integers has one flux quantum per plaquette. If the bosons are restricted to this lattice, given the analytic structure of (1) each site can only have boson occupancies 0, 1, and 2, and, because the filling factor is $\nu = 1$, there will be an average of one boson per site.

The spin-one CSL state constructed using (1) is [16]

$$|\Psi\rangle = \sum_{z_1,\ldots,z_N}\Psi[z_i]\prod_{i}^{N} G(z_i)\tilde{S}^{+}_{z_1}\ldots\tilde{S}^{+}_{z_N}|-1\rangle_N, \qquad (2)$$

where the z_i's are summed over all lattice points η_{nm}. Here $G(\eta_{nm}) = (-1)^{(n+1)(m+1)}$ is a gauge phase, and the operators $\tilde{S}^{+}_{z}$ are renormalized spin-flip operators,

$$\tilde{S}^{+}_{\alpha} = \tfrac{1}{2}\left(S^{z}_{\alpha} + 1\right)S^{+}_{\alpha}, \qquad (3)$$

acting on the state $|-1\rangle_N = \otimes_{\alpha=1}^{N}|1,-1\rangle_\alpha$ in which a spin-one in the $S_z = -1$ state (i.e., 0 boson occupancy) sits on each site. Both the gauge phase and spin-flip operators are chosen so that $|\Psi\rangle$ becomes a singlet in the thermodynamic limit [16,18].

A similar state was studied in Ref. [17] that, in the large system limit, becomes identical to that proposed in Ref. [16] for the planar geometry.

When this construction is generalized to the torus the CSL states are again of the form (2), but there is now a three-dimensional space of states corresponding to the threefold topologically degeneracy of the bosonic Moore-Read states on the torus. For a rectangular $L_x \times L_y$ system in the Landau gauge this space is spanned by the states [19,20]

$$\Psi_\alpha[z_i] = \mathrm{Pf}\left(\frac{\vartheta_{\alpha+1}((z_i - z_j)/L_x|\tau)}{\vartheta_1((z_i - z_j)/L_x|\tau)}\right) \times \prod_{i<j}^{N} \vartheta_1((z_i - z_j)/L_x|\tau) F_{\mathrm{cm}}^{(\alpha)}(Z) \prod_{i=1}^{N} e^{-y_i^2/2}, \quad (4)$$

where

$$\vartheta_\delta(z|\tau) = (-1)^{\tilde{\delta}} \sum_{n=-\infty}^{\infty} e^{[i\pi\tau(n+a)^2 + 2\pi i(n+a)(z+b)]} \quad (5)$$

are the four Jacobi theta functions where the parameters (a,b) take the values $(1/2,1/2),(1/2,0),(0,0),(0,1/2)$, for $\delta = 1,2,3,4$, respectively, and $\tilde{\delta} = 1$ only for $\delta = 1$, otherwise $\tilde{\delta} = 0$. The parameter τ is determined by the ratio of the system lengths $\tau = iL_y/L_x$. As above, the lattice of points $z = \eta_{nm} = \sqrt{2\pi}(n + im)$ has one flux quantum per plaquette, and, again, when bosons are confined to this lattice the allowed occupancies are 0, 1, and 2. Finally, for even by even lattices the center-of-mass term $F_{\mathrm{cm}}^{(\alpha)}(Z)$, where $Z = \sum_{i=1}^{N} z_i$, is taken to be

$$F_{\mathrm{cm}}^{(\alpha)}(Z) = \vartheta_{\alpha+1}(Z/L_x|\,\tau), \quad (6)$$

to ensure the wave function is periodic for each boson on the lattice with period L_x (L_y) in the x (y) direction.

The torus CSL states are again constructed using (2) but with Ψ replaced by one of the three Ψ_α states defined above and with a new gauge factor $G(z_i)$ with $G(\eta_{n,m}) = (-1)^{(n+m)}$ which takes into account the change from symmetric to Landau gauge. On the torus, the resulting CSL states $|\Psi_\alpha\rangle$ are exact singlets, even for finite systems [18]. This procedure generalizes the Abelian CSL construction on the torus due to Laughlin [21]. These torus Abelian states were studied by Monte Carlo similar to that used here in Ref. [22]. A general prescription for constructing torus CSL states based on conformal field theory, which includes the non-Abelian case relevant here, was given in Ref. [23].

We have carried out Monte Carlo calculations for both the droplet and torus CSL states. In all cases the Pfaffian becomes singular when two bosons occupy the same site. However, the wave function remains finite, because the corresponding Jastrow factor "cancels" the divergence of the Pfaffian. In our simulations we treat this singular case by replacing the relevant Jastrow factor and Pfaffian element with 1 for any doubly occupied site, thus correctly reproducing the limiting value of their product.

Correlations. Figure 1 shows spin correlation functions $\langle \mathbf{S}_0 \cdot \mathbf{S}_{n_x}\rangle$ and $\langle S_0^z S_{n_x}^z\rangle$ for the droplet CSL where 0 is the droplet center and $n_x = x/\sqrt{2\pi}$ is the number of lattice spacings along the x direction. Results are shown for $N = 100$ and 180 bosons

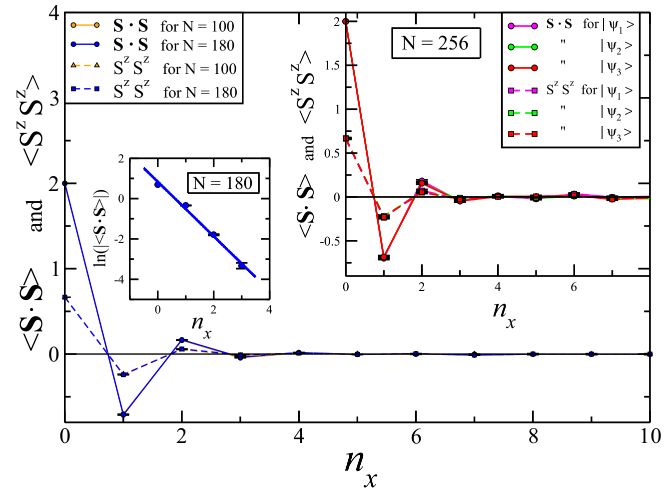

FIG. 1. Spin correlation functions $\langle \mathbf{S}_0 \cdot \mathbf{S}_{0+n_x}\rangle$ and $\langle S_0^z S_{0+n_x}^z\rangle$ versus n_x (lattice spacings in x direction) for droplet CSL states with $N = 100$ and 180 bosons (0 is the droplet center). The left inset shows a logarithmic plot of $|\langle \mathbf{S}_0 \cdot \mathbf{S}_{0+n_x}\rangle|$ with linear fit yielding a correlation length of $\xi = 1.35 \pm 0.14$. The right inset shows the spin correlation functions $\langle \mathbf{S}_i \cdot \mathbf{S}_{i+n_x}\rangle$ and $\langle S_i^z S_{i+n_x}^z\rangle$ for the three states $|\Psi_\alpha\rangle$ on a toroidal lattice of size 16×16. The spin correlations in these states are indistinguishable within errors.

and it is evident that the correlations for the different system sizes agree. Note that $\langle S_0^z S_{n_x}^z\rangle \simeq \frac{1}{3}\langle \mathbf{S}_0 \cdot \mathbf{S}_{n_x}\rangle$ consistent with the approximate singlet nature of the droplet CSL. We find the absolute value of the spin correlation functions follow a simple exponential law, $|\langle \mathbf{S}_0 \cdot \mathbf{S}_{n_x}\rangle| \propto e^{-n_x/\xi}$, even at short distance, consistent with the expectation that the spin-one CSL can be viewed as a gapped spin liquid. From our numerics we obtain a spin correlation length of $\xi = 1.35 \pm 0.14$ lattice spacings (see Fig. 1 left inset).

Figure 1 also shows spin correlation functions for all three CSL states $|\Psi_\alpha\rangle$ on the torus for a 16×16 lattice. Our results confirm that for a large enough system these correlation functions coincide for all three states within errors (see Fig. 1 right inset), and also agree with the droplet correlations. We note that this is not the case for small system sizes. For example, for the simple case of a 2×2 torus all correlation functions can be obtained analytically for all three states with clearly distinguishable results [see Supplemental Material (SM) [24]].

One difference between the droplet and torus CSL states, noted above, is that the droplet only becomes an exact singlet in the thermodynamic limit. We can see this explicitly by noting that for a singlet state the onsite correlations must satisfy $\langle S_i^z S_i^z\rangle = \frac{1}{2}\langle S_i^+ S_i^-\rangle = \frac{1}{2}\langle S_i^- S_i^+\rangle = \frac{1}{3}\langle \mathbf{S}_i^2\rangle = \frac{2}{3}$. For the case of a CSL droplet with four bosons we find that, at the droplet center, $\langle S_0^z S_0^z\rangle \approx 0.72$ and $\frac{1}{2}\langle S_0^{+(-)} S_0^{-(+)}\rangle \approx 0.86(0.42)$. However, for droplets of 20 bosons or more, all three correlations have nearly converged to the singlet value of $\frac{2}{3}$. In contrast, for the torus our numerics confirm that, even for small system sizes, the expectation values $\langle S_i^z S_i^z\rangle$ and $\frac{1}{2}\langle S_i^{+(-)} S_i^{-(+)}\rangle$ are precisely $\frac{2}{3}$ on all sites. The fact that the value of these onsite correlation functions provide a nontrivial test of the singlet nature of the

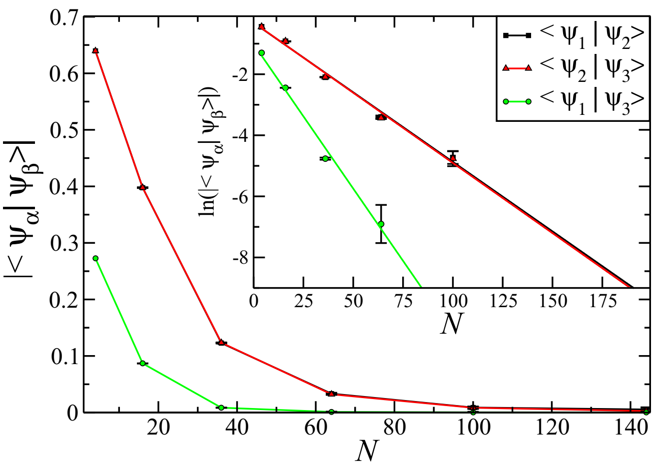

FIG. 2. Normalized overlaps $|\langle\Psi_{1(2)}|\Psi_{2(3)}\rangle|$ and $|\langle\Psi_1|\Psi_3\rangle|$ for square-shaped systems versus number of lattice sites $N = 4,16,36,64,100,144$. The inset shows logarithmic plots of $|\langle\Psi_{1(2)}|\Psi_{2(3)}\rangle|$ and $|\langle\Psi_1|\Psi_3\rangle|$ versus N with linear fits showing $|\langle\Psi_{1(2)}|\Psi_{2(3)}\rangle|$ becomes exponentially smaller with a decay factor of $\zeta = 0.05 \pm 0.01$, while $|\langle\Psi_1|\Psi_3\rangle|$ decreases with $\zeta = 0.095 \pm 0.001$.

spin-one CSL can be contrasted with the spin-1/2 case for which $\langle S_i^z S_i^z\rangle$ is always equal to $\frac{1}{4}$.

Orthogonality. To establish that the three torus CSL states $|\Psi_\alpha\rangle$ (henceforth assumed normalized) span a three-dimensional space we have calculated their overlap matrix for several square-shaped lattices of sizes $2\times2,\dots,12\times12$. In all cases we find the overlap matrix has full rank. Moreover, the off-diagonal matrix elements go to zero exponentially as $e^{-\zeta N}$ where N is the number of lattice sites, with $\zeta = 0.05 \pm 0.01(0.095 \pm 0.001)$ for $|\langle\Psi_1|\Psi_3\rangle|$ ($|\langle\Psi_1|\Psi_2\rangle|$ and $|\langle\Psi_2|\Psi_3\rangle|$), as shown in Fig. 2. Thus, the three states become orthogonal in the thermodynamic limit. More details are given in the SM [24].

The transformation properties of theta functions under modular transformations imply that, for square-shaped systems, $R_{\pi/2}|\Psi_{1,3}\rangle = |\Psi_{3,1}\rangle$ and $R_{\pi/2}|\Psi_2\rangle = |\Psi_2\rangle$, where $R_{\pi/2}$ generates a $\pi/2$ rotation in the plane. We therefore expect $|\langle\Psi_1|\Psi_2\rangle| = |\langle\Psi_2|\Psi_3\rangle|$ for any square-shaped system as the numerical results in Fig. 2 confirm. These symmetry properties are also apparent in the 2×2 spin correlation functions given in Table I in the SM [24].

Entanglement entropy. The three states $|\Psi_\alpha\rangle$ become orthogonal and possess indistinguishable spin correlations in the thermodynamic limit. This threefold topological degeneracy is consistent with the natural hypothesis that the spin-one CSL state, like the bosonic Moore-Read state on which it is based, is described by $SU(2)_2$ Chern-Simons theory [25,26]. To provide further evidence that this is the case we turn to the entanglement entropy.

The Renyi entropy of order n associated with a partitioning of the system into a region A and its compliment B is defined as $S_n = -\frac{1}{n-1}\ln \mathrm{Tr}(\rho_A^n)$, where $\rho_A = \mathrm{Tr}_B|\Psi\rangle\langle\Psi|$ is the reduced density matrix of region A. Ground states of gapped local Hamiltonians exhibit a boundary law scaling which can generically be written in two dimensions for simply-connected regions A as $S_n(\rho_A) = \alpha_n L_A - \gamma + \cdots$. The leading term is proportional to L_A, the boundary length of region A, while the second term, $-\gamma$, is the topological entanglement entropy (TEE), characteristic of topological phases [27,28].

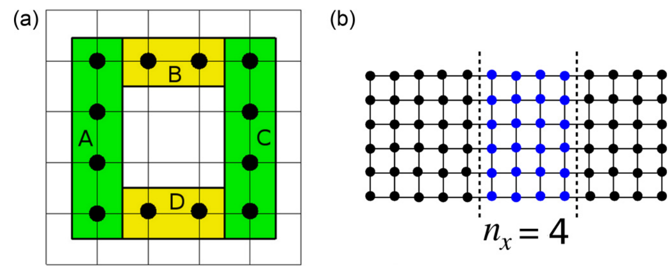

FIG. 3. (a) Example regions A, B, C, and D, used in the Levin-Wen construction to isolate the TEE. (b) 12×6 torus with dashed lines indicating an example region for which the Renyi entropy S_2 is calculated when partitioning the toroidal system into two cylinders.

In a topologically ordered state the TEE is determined by the total quantum dimension $\mathcal{D}$, $\gamma = \ln\mathcal{D}$, where $\mathcal{D}$ is defined through the quantum dimension d_i of the quasiparticles of the underlying topological field theory: $\mathcal{D} = \sqrt{\sum_i d_i^2}$. For the spin-one CSL, based on the continuum bosonic Moore-Read state, we expect the $SU(2)_2$ quantum dimensions of $1,1,\sqrt{2}$ for which $D = 2$ and $\gamma = \ln 2$.

We proceed by calculating the $n = 2$ Renyi entropy using the replica method [29]. Details are given in the SM [24]. One way to isolate the TEE is to employ the Levin-Wen [27] construction [see Fig. 3(a)], where the area-dependent part cancels from a superposition of four entropies: $-2\gamma = (S_{ABCD} - S_{ADC}) - (S_{ABC} - S_{AC})$. To combat large error bars, we employed the reweighting scheme of Ref. [30] (see SM [24]). We first choose a relatively small system of size 6×6 and Levin-Wen regions A,B,C and D as shown in Fig. 3(a), resulting in $\gamma = 1.16 \pm 0.08(1.14 \pm 0.08, 1.04 \pm 0.07)$ for the states $|\Psi_{1(2,3)}\rangle$. The value is above the theoretically expected $\ln 2 \approx 0.69$, but upon increasing the system size to 8×8, with regions $A(C)$ of size 1×6 and $B(D)$ of size 3×2, we find $\gamma = 0.91 \pm 0.32$ for $|\Psi_1\rangle$, consistent with γ approaching $\ln 2$ in the thermodynamic limit. This is also consistent with the result for γ obtained numerically in Ref. [17] using a bicylindrical cut of a CSL state on the cylinder with open boundary conditions.

To identify the modular $\mathcal{S}$ matrix associated with the topological field theory describing the CSL state we follow Ref. [31] and let $|\Xi_i\rangle$ denote the $\hat{y}$ direction Wilson loop eigenstates associated with quasiparticle of quantum dimension d_i for $i = 1,2,3$. The overlap matrix $V_{ij} = \langle\Xi_i|R_{\pi/2}|\Xi_j\rangle$ between the (normalized) bases $\{|\Xi_i\rangle\}$ and $\{R_{\pi/2}|\Xi_j\rangle\}$ (the $\hat{x}$ direction Wilson loop states) is then related to the modular $\mathcal{S}$ matrix by $V = D^\dagger \mathcal{S} D$, where D is a diagonal matrix of phases $D_{jj} = e^{i\Phi_j}$ corresponding to the phase freedom of choosing $|\Xi_j\rangle$. It follows that the eigenvalues $R_{\pi/2}$ are the same as those of the modular $\mathcal{S}$ matrix.

As noted above, for square-shaped systems, $R_{\pi/2}|\Psi_{1,3}\rangle = |\Psi_{3,1}\rangle$ and $R_{\pi/2}|\Psi_2\rangle = |\Psi_2\rangle$. This, together with the fact that the $|\Psi_\alpha\rangle$ states become orthogonal for large systems, implies the eigenvalues of $\mathrm{R}_{\pi/2}$ are $\{1,1,-1\}$. The $\mathcal{S}$ matrix for $SU(2)_2$ Chern-Simons theory has the same set of eigenvalues

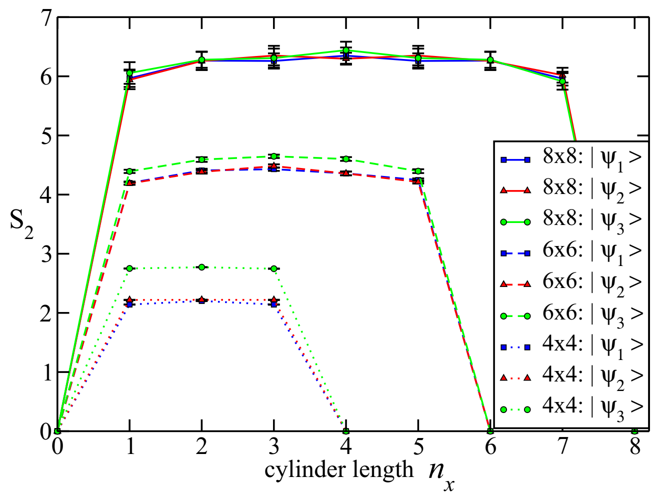

FIG. 4. S_2 versus the length of the cylindrical region A for square-shaped systems for the three CSL states $|\Psi_\alpha\rangle$. On a 6×6 lattice, two wave functions, $|\Psi_{1,2}\rangle$, have identical S_2 within error bars. For an 8×8 lattice, S_2 is the same for all three ground states within error bars.

and is the only such rank 3 $\mathcal{S}$ matrix [32]. Thus, if the spin-one CSL is described by a topological field theory it *must* have quasiparticles with quantum dimensions $d_{1,2}=1$ and $d_3=\sqrt{2}$.

To connect this observation to our numerics, we note that for such a topologically ordered state the TEE becomes state dependent when S_2 is calculated on the torus over a (non-simply-connected) cylindrical region of length n_x such as that shown in Fig. 4(b) [26,31], with $S_2=-\gamma'+\alpha_2 L_A$, where

$$\gamma' = 2\gamma + \ln\left(\sum_j p_j^2/d_j^2\right). \qquad (7)$$

Here, $p_j=|c_j|^2$ where $|\Psi_\alpha\rangle=\sum_j c_j|\Xi_j\rangle$. We have numerically calculated S_2 for all three torus CSL states on square-shaped lattices up to size 8×8. The results are shown in Fig. 4. We observe first that S_2 saturates as n_x increases [for $n_x<\frac{1}{2}L_x/(\sqrt{2\pi})$], consistent with these states being possible ground states of a gapped Hamiltonian. Further we find that for large enough systems S_2 is the same for all three states $|\Psi_\alpha\rangle$, and thus γ' is as well.

The observation that γ' is state independent for the $|\Psi_\alpha\rangle$ states, together with the requirement that the eigenvectors with eigenvalue -1 of the known $\mathcal{S}$ matrix for $SU(2)_2$ [26,32,33] (for the the phase choice $\Phi_j=0$ for $j=1,2,3$), $(|\Xi_1\rangle-|\Xi_2\rangle)/2-|\Xi_3\rangle/\sqrt{2}$, and of $R_{\pi/2}$, $(|\Psi_1\rangle-|\Psi_3\rangle)/\sqrt{2}$, must be the same (up to a phase), constrains us to make the identification,

$$|\Psi_{2,a}\rangle = \frac{1}{\sqrt{2}}(|\Xi_1\rangle \pm |\Xi_2\rangle), \quad |\Psi_b\rangle = |\Xi_3\rangle\,, \qquad (8)$$

where $(|\Psi_a\rangle,|\Psi_b\rangle)=(|\Psi_1\rangle,|\Psi_3\rangle)$ or $(|\Psi_3\rangle,|\Psi_1\rangle)$. For both choices it is readily seen that if quasiparticles with $d_{1,2}=1$ are associated with $|\Xi_{1,2}\rangle$ and the non-Abelian excitation with $d_3=\sqrt{2}$ is associated with $|\Xi_3\rangle$, (7) does indeed yield $\gamma'=\ln 2$ for *all* three states $|\Psi_\alpha\rangle$. Our numerical observation that γ' is the same for the states $|\Psi_\alpha\rangle$ is thus consistent with these states being identified as a basis for the three-dimensional topological Hilbert space of an $SU(2)_2$ Chern-Simons theory on the torus.

Conclusion. In this paper, we investigated several properties of a spin-one CSL on the square lattice. Spin correlations were found to decay exponentially, and, for the torus, become indistinguishable for the states $|\Psi_\alpha\rangle$ for large systems. We further found these states become orthogonal in the thermodynamic limit.

A Levin-Wen construction was used to determine the TEE of the CSL with results consistent with $-\ln 2$ in the thermodynamic limit. In addition, based purely on symmetry, we argued that the modular $\mathcal{S}$ matrix of the CSL (if it exists) must be the same as that for the bosonic Moore-Read state. These observations, together with the observation that for large enough systems the cylindrical entropies for the states $|\Psi_\alpha\rangle$ are all the same, are consistent with the spin-one CSL exhibiting the non-Abelian topological order of $SU(2)_2$ Chern-Simons theory.

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[1] X.-G. Wen, Int. J. Mod. Phys. B **4**, 239 (1990).
[2] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991).
[3] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Rev. Mod. Phys. **80**, 1083 (2008).
[4] S. D. Sarma, M. Freedman, and C. Nayak, npj Quantum Information **1**, 15001 (2015).
[5] P. W. Anderson, Mater. Res. Bull. **8**, 153 (1973).
[6] A. Seidel, Phys. Rev. B **80**, 165131 (2009).
[7] J. Wildeboer and A. Seidel, Phys. Rev. Lett. **109**, 147208 (2012).
[8] J. Wildeboer, A. Seidel, and R. G. Melko, arXiv:1510.07682.
[9] N. Schuch, D. Poilblanc, J. I. Cirac, and D. Pérez-García, Phys. Rev. B **86**, 115108 (2012).
[10] Z. Zhou, J. Wildeboer, and A. Seidel, Phys. Rev. B **89**, 035123 (2014).
[11] V. Kalmeyer and R. B. Laughlin, Phys. Rev. Lett. **59**, 2095 (1987).
[12] V. Kalmeyer and R. B. Laughlin, Phys. Rev. B **39**, 11879 (1989).
[13] D. F. Schroeter, E. Kapit, R. Thomale, and M. Greiter, Phys. Rev. Lett. **99**, 097202 (2007).
[14] R. Thomale, E. Kapit, D. F. Schroeter, and M. Greiter, Phys. Rev. B **80**, 104406 (2009).
[15] M. Greiter, D. F. Schroeter, and R. Thomale, Phys. Rev. B **89**, 165125 (2014).
[16] M. Greiter and R. Thomale, Phys. Rev. Lett. **102**, 207203 (2009).

[17] I. Glasser, J. I. Cirac, G. Sierra, and A. E. B. Nielsen, New J. Phys. **17**, 082001 (2015).
[18] M. Greiter and R. Thomale, Phys. Rev. B **85**, 155145 (2012).
[19] M. Greiter, X. Wen, and F. Wilczek, Nucl. Phys. B **374**, 567 (1992).
[20] N. Read and E. Rezayi, Phys. Rev. B **54**, 16864 (1996).
[21] R. Laughlin, Ann. Phys. (NY) **191**, 163 (1989).
[22] N. E. Bonesteel, Phys. Rev. A **62**, 062310 (2000).
[23] A. E. Nielsen and G. Sierra, J. Stat. Mech.: Theor. Exp. (2014) P04007.
[24] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.94.045125 for details on how the wave function overlaps were obtained by Monte Carlo and for further information concerning the entanglement entropy and related Monte Carlo calculations in a non-Abelian spin-one spin liquid.
[25] E. Fradkin, C. Nayak, A. Tsvelik, and F. Wilczek, Nucl. Phys. B **516**, 704 (1998).
[26] S. Dong, E. Fradkin, R. G. Leigh, and S. Nowling, J. High Energy Phys. 05 (2008) 016.
[27] M. Levin and X.-G. Wen, Phys. Rev. Lett. **96**, 110405 (2006).
[28] A. Kitaev and J. Preskill, Phys. Rev. Lett. **96**, 110404 (2006).
[29] M. B. Hastings, I. González, A. B. Kallin, and R. G. Melko, Phys. Rev. Lett. **104**, 157201 (2010).
[30] J. Pei, S. Han, H. Liao, and T. Li, J. Phys. Condens. Matter **26**, 035601 (2014).
[31] Y. Zhang, T. Grover, A. Turner, M. Oshikawa, and A. Vishwanath, Phys. Rev. B **85**, 235151 (2012).
[32] E. Rowell, R. Stong, and Z. Wang, Commun. Math. Phys. **292**, 343 (2009).
[33] J. L. Cardy, Nucl. Phys. B **270**, 186 (1986).
[34] M. Troyer, B. Ammon, and E. Heeb, Lect. Notes Comput. Sci. **1505**, 0191 (1998).
[35] A. Albuquerque, F. Alet, P. Corboz, P. Dayal, A. Feiguin, S. Fuchs, L. Gamper, E. Gull, S. Gürtler, A. Honecker *et al.*, J. Magn. Magn. Mater. **310**, 1187 (2007).