Valence bonds and the Lieb-Schultz-Mattis theorem

N. E. Bonesteel
Department of Physics, The Ohio State University, 174 West 18th Avenue, Columbus, Ohio 43210
(Received 19 May 1989)

The decoupling of short-range valence-bond states into topologically distinct sectors is discussed. On a square-lattice strip with odd width, this decoupling implies that short-range resonating-valence-bond (RVB) states are doubly degenerate. This result is precisely equivalent to the Lieb-Schultz-Mattis (LSM) theorem. Consequently short-range RVB states are consistent with the LSM theorem whether or not they possess gapless excitations. In the limit of infinite strip width, the decoupling becomes fourfold, in agreement with a recent conjecture of Haldane.

I. INTRODUCTION

Any singlet state defined on a lattice of spin-$\frac{1}{2}$ objects may be represented as a linear superposition of valence-bond states.[1] In a valence-bond state spins are singlet correlated in pairs and are said to be connected by valence bonds. Although the valence-bond state basis is both overcomplete and nonorthogonal, it provides a natural language with which to discuss the properties of singlet states on lattices. Recent interest has focused on spin liquids[2]—singlet states that exhibit no long-range Néel order nor any other form of locally observable broken symmetry.[3] A spin-liquid state in which mobile holes have been introduced may provide a theoretical description of the superconducting state in high-T_c materials.[2,4] One type of spin liquid, known as a short-range resonating-valence-bond (RVB) state,[5,4,6] is constructed by superimposing valence-bond states with only bonds of short range, i.e., bonds which do not exceed a given length l_0,[7] in such a way as to produce a featureless, liquidlike state. Short-range RVB states are believed to have both exponentially decaying spin correlations and a gap to spin-1 excited states.[8,9]

A theorem of Lieb, Schultz, and Mattis (LSM) (Ref. 10) states that in one dimension a class[11] of translationally invariant spin-$\frac{1}{2}$ antiferromagnetic Hamiltonians cannot have both a unique ground state and an energy gap. In particular, one possibility allowed by the theorem is a doubly degenerate ground state separated by a gap from the excited states. In this case the ground-state manifold must be spanned by two states, $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$, with the property that $T|\psi_\alpha\rangle=|\psi_\beta\rangle$ and $T|\psi_\beta\rangle=|\psi_\alpha\rangle$, where T is an operator translating the chain through one lattice vector. Also, if we define the LSM "slow-twist" operator

$$U=\exp\left[i(2\pi/L)\sum_{X=1}^{L} XS_X^z\right], \tag{1}$$

where L is the length of the periodic chain, then, in the limit $L\to\infty$: $U|\psi_\alpha\rangle=+|\psi_\alpha\rangle$ and $U|\psi_\beta\rangle=-|\psi_\beta\rangle$. Affleck[12] has recently obtained the same result for infinite strips with odd width using an appropriately generalized slow-twist operator. In one dimension it has been shown that if there is an energy gap the ground-state degeneracy required by the LSM theorem is due to a locally observable broken translational symmetry.[13] Thus the only type of spin-$\frac{1}{2}$ spin-liquid state which can exist in one dimension must have gapless spin excitations, i.e., it must resemble Bethe's solution to the nearest-neighbor Heisenberg model which can be visualized as a superposition of valence-bond states containing bonds on all length scales—not just short bonds. In this paper we address the following question: Is Affleck's two-dimensional (2D) generalization of the LSM theorem consistent with a short-range RVB state having both an energy gap and no locally observable broken symmetry? The answer is yes.

Early on in the development of RVB theory it was recognized that short-range valence-bond configurations decouple into topologically distinct sectors.[14–16] The topological quantum numbers labeling the different sectors arise purely because valence bonds connect two sites and only one bond may be attached to each site. Rokhsar and Kivelson[16] have pointed out that one consequence of this decoupling is that on an odd-width square-lattice strip any translationally invariant singlet state made up of valence-bond states in which bonds connect only neighboring sites must be doubly degenerate.[17] In this paper their observation is extended to arbitrary short-range valence-bond states with the same result. Also, it is argued that the requirement of this degeneracy is equivalent to the LSM theorem; i.e., that it is in fact the *same* theorem. A consequence of this equivalence is that short-range RVB states are consistent with the LSM theorem. Finally, it is shown that in the limit of infinite strip width there is a fourfold topological decoupling of short-range valence-bond states, in agreement with a recent conjecture of Haldane.[18]

An outline of this paper follows. In Sec. II the LSM theorem and its generalization to odd-width strips are reviewed. In Sec. III we discuss the topological properties of short-range valence-bond states and the connection between these properties and the LSM theorem. Section IV describes the implications of the LSM theorem and valence-bond structure for short-range RVB states, and it is argued that on odd-width strips such states will exhibit a locally observable broken translation symmetry which becomes *unobservable* in the limit of infinite strip width. In Sec. V the implications of the topological decoupling for $L\times L$ lattices with periodic boundary conditions are considered. In particular we argue that short-range RVB states will be fourfold degenerate with no locally observable broken symmetry. Finally, Sec. VI summarizes the main results of this paper.

II. THE LIEB-SCHULTZ-MATTIS THEOREM

A brief review of the LSM theorem follows; for a more complete discussion see Refs. 12 and 13. First consider one dimension and a specific Hamiltonian:

$$H=J\sum_{i=1}^{L}(\mathbf{S}_i\cdot\mathbf{S}_{i+1}+\beta\mathbf{S}_i\cdot\mathbf{S}_{i+2}) \tag{2}$$

defined on a periodic chain of length L. Imagine that H has a unique ground state $|\psi_0\rangle$. A state, $|\psi_1\rangle$, can be constructed which is orthogonal to $|\psi_0\rangle$ and which satisfies

$$\langle\psi_1|(H-E_0)|\psi_1\rangle\rightarrow 0$$

in the limit $L\rightarrow\infty$, where E_0 is the ground-state energy. This is done by acting on $|\psi_0\rangle$ with the LSM slow-twist operator U defined in (1): $|\psi_1\rangle=U|\psi_0\rangle$. It is straightforward to show that U and $|\psi_1\rangle$ have the following properties:

$$TU=-UT\ , \tag{3a}$$

$$\langle\psi_1|(H-E_0)|\psi_1\rangle=J\left[[\cos(2\pi/L)-1]\sum_{i=1}^{L}\langle\psi_0|S_i^+S_{i+1}^-|\psi_0\rangle+\beta[\cos(4\pi/L)-1]\sum_{i=1}^{L}\langle\psi_0|S_i^+S_{i+2}^-|\psi_0\rangle\right]$$
$$\sim O(J/L)\ . \tag{3b}$$

It has been assumed that $|\psi_0\rangle$ is the unique ground state of H; hence it must be an eigenstate of T. An arbitrary phase factor in the definition of T may be fixed by insisting that $T|\psi_0\rangle=|\psi_0\rangle$, i.e., $|\psi_0\rangle$ has a crystal momentum of 0. Equation (3a) then implies that $T|\psi_1\rangle=-|\psi_1\rangle$, i.e., $|\psi_1\rangle$ has a crystal momentum of π, and is thus orthogonal to $|\psi_0\rangle$. Equation (3b) implies that $|\psi_1\rangle$ has energy $E_0+O(J/L)$ so that in the $L\rightarrow\infty$ limit $|\psi_0\rangle$ and $|\psi_1\rangle$ become degenerate. There are then two possibilities: Either $|\psi_1\rangle$ is a gapless excited state or there is an energy gap with the ground state being at least doubly degenerate.

Consider the latter possibility: a doubly degenerate ground state with a gap to all other excited states. This is known[19] to be the case for the Hamiltonian defined in (2) when $\beta=\frac{1}{2}$, and is believed[20] to be true over a wide range of β values. The ground-state manifold is then entirely spanned by $|\psi_0\rangle$ and $|\psi_1\rangle$, both of which are eigenstates of the translation operator. Now define

$$|\psi_\alpha\rangle=(\tfrac{1}{2})^{1/2}(|\psi_0\rangle+|\psi_1\rangle)$$

and

$$|\psi_\beta\rangle=(\tfrac{1}{2})^{1/2}(|\psi_0\rangle-|\psi_1\rangle)\ .$$

The states $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$ span the ground-state manifold and are eigenstates of U with eigenvalues $+1$ and -1, respectively. Furthermore, $T|\psi_\alpha\rangle=|\psi_\beta\rangle$ and $T|\psi_\beta\rangle=|\psi_\alpha\rangle$. Because $|\psi_\alpha\rangle$ is a ground state of H but not an eigenstate of T, it appears that translational symmetry has been spontaneously broken. Affleck and Lieb[13] proved that this was in fact the case by explicitly constructing a local operator, i.e., an operator which only acts on a finite number of spins in the $L\rightarrow\infty$ limit, which "measured" this broken symmetry.

The LSM theorem can be extended to frustrated spin-$\frac{1}{2}$ antiferromagnetic Hamiltonians defined on odd-width square-lattice strips.[12] The two-dimensional slow-twist operator

$$U_{2D}=\exp\left[i(2\pi/L)\sum_{X=1}^{L}X\sum_{Y=1}^{M}S_{X,Y}^z\right] \tag{4}$$

satisfies (3a) as long as M is odd. Thus acting on a "unique" ground state $|\psi_0\rangle$ with U_{2D} will again produce an orthogonal state $|\psi_1\rangle$, this time with energy $E_0+O(JM/L)$. In the limit of infinite strip length $|\psi_1\rangle$ is either a gapless excited state or a degenerate ground state. Again, if the latter case is realized one can define $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$ as above: Two orthogonal vectors which span the ground-state manifold are eigenvectors of U_{2D} with eigenvalues $+1$ and -1, respectively, and satisfy $T|\psi_\alpha\rangle=|\psi_\beta\rangle$ and $T|\psi_\beta\rangle=|\psi_\alpha\rangle$. Again $|\psi_\alpha\rangle$ is a ground state of H but not an eigenstate of T, suggesting a broken translational symmetry. However, in this case, one cannot construct a local operator which is guaranteed to measure the broken symmetry in the $M\rightarrow\infty$ limit.[12] Thus it is possible that the broken symmetry becomes unobservable in this limit. We will argue that this is in fact the case for the short-range RVB states.

III. VALENCE-BOND TOPOLOGY

Because of the constraint that valence bonds connect two sites and that only one bond may be attached to each site, short-range valence-bond states possess global topological "quantum numbers," i.e., properties which are unaffected by local changes in the bond configuration. There are at least two equivalent ways of describing the global properties of short-range valence-bond states: the winding number introduced by Kivelson, Rokhsar, and Sethna[14,16] and the gap parity introduced by Thouless.[15] Both schemes are reviewed in the following. The gap-parity scheme is then used to show that short-range valence-bond states on odd-width strips decouple into two sectors we denote (e,o) and (o,e), and that a (nonrigorous) consequence of this is that any translationally invariant state that may be written as a superposition of short-range valence-bond states will be doubly degenerate.

Figures 1(a) and 1(b) depict two short-range valence-bond configurations realized on 6×6 lattices with periodic boundary conditions. One configuration is an arbitrary short-range valence-bond configuration [Fig. 1(a)]; the other is a "column" configuration [Fig. 1(b)]. The x and y projection of the length of any bond in Fig. 1(a) never exceeds two, so that the way in which a given bond wraps around the torus can be unambiguously specified. When these configurations are overlayed, a "transition graph" is produced [Fig. 1(c)]. Such a transition graph may be associated with any short-range valence-bond configuration by overlaying it with a reference column state. Each transition graph is made up entirely of closed loops. Because of the periodic boundary conditions, nontrivial loops which encircle the torus in either the x,y or both directions may occur in this graph. The winding numbers Ω_x and Ω_y are then defined to be the number of times such loops encircle the torus in the x direction and y direction, respectively. One can readily convince oneself that any local change in the configuration shown in Fig. 1(a) may change Ω_x and Ω_y, but only by integer multiples of 2. Thus Ω_x and Ω_y modulo two are topological invariants.

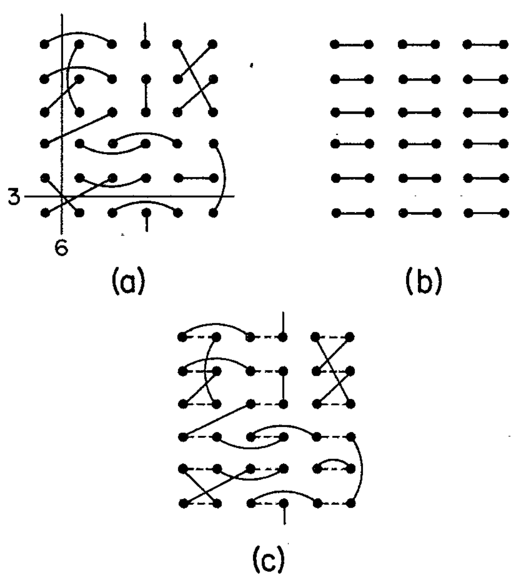

FIG. 1. Two short-range valence-bond configurations and their transition graph, all realized on 6×6 lattices with periodic boundary conditions. Configuration (a) is an arbitrary short-range valence-bond state. Two solid lines, one vertical and one horizontal, cut through gaps in the lattice. The vertical line cuts three bonds and the horizontal line cuts six. Any vertical line slicing the configurations will cut an odd number of bonds and any horizontal line will cut an even number of bonds. Local changes in the bond configuration will not alter the parity of the number of bonds cut by horizontal and vertical lines. Configuration (b) is a reference column configuration which when overlayed with (a) produces (c), the transition graph. The transition graph contains closed loops consisting of alternating (a) (solid) and (b) (dashed) bonds. The winding numbers of (a) are defined as follows: Ω_x and Ω_y are equal to the net number of times loops wrap around the x and y directions, respectively. One can readily convince oneself that any local change in configuration (a) will only change Ω_x or Ω_y by integer multiples of 2. In the example shown here $\Omega_x=0$ and $\Omega_y=1$.

In Fig. 1(a) two solid lines cut through gaps in the lattice, one vertically and one horizontally. The vertical line crosses three valence bonds while the horizontal line crosses six. A line cutting through any of the vertical gaps will cross an odd number of bonds and a line cutting through any of the horizontal gaps will cross an even number. As with the winding number any local change in the bond configuration will leave the parity of the bonds in the horizontal and vertical gaps unchanged and so the gap parities are topological invariants. Not surprisingly it can be shown that the gap parity and the winding number schemes are equivalent. Any two valence-bond configurations with the same winding numbers (modulo two) will have the same gap parities and vice versa. A consequence of this is that the transition graph produced by overlaying two configurations with different gap parities will invariably contain nontrivial loops.

Now consider an $M\times L$ lattice with M odd. Rokhsar and Kivelson[16] have pointed out that on such a lattice the decoupling of short-range valence-bond states into topologically distinct sectors implies that any state which can be represented as a superposition of nearest-neighbor valence-bond states must be doubly degenerate. They used the winding number classification scheme. Here we use the gap-parity scheme to extend their observation to arbitrary short-range valence-bond states and clarify the connection of this degeneracy with that required by the LSM theorem.

Figure 2 depicts a short-range valence-bond configuration realized on a 5×10 lattice. In the figure the lattice only has periodic boundary conditions in the x direction, though this is not crucial for what follows. The x component of each bond does not exceed length four. By slicing the configuration as shown in the figure and counting the number of bonds cut by each slice, one finds an alternating even-odd pattern. Again, local changes in the bond configuration will not alter this pattern; it will occur for *any* bond configuration no matter how random as long as M is odd. Thus short-range valence-bond states defined on odd-width strips fall into two classes we call even-odd (e,o) and odd-even (o,e). Because of the alternating property of these configurations the translation operator T has the effect of mapping states belonging to (e,o) into (o,e) and vice versa.

Some properties of short-range valence-bond states and the (e,o), (o,e) decoupling are now established. In the limit $L\to\infty$, M fixed and odd, any two valence-bond states $|\alpha\rangle\in(o,e)$ and $|\beta\rangle\in(e,o)$ have the following properties: (i) $\langle\alpha|\beta\rangle=0$; (ii) for any Hamiltonian H containing only short-range spin interactions, $\langle\alpha|H|\beta\rangle=0$; and (iii) for the two-dimensional generalization of the LSM slow-twist operator $U_{2D}|\alpha\rangle=+|\alpha\rangle$ and $U_{2D}|\beta\rangle=-|\beta\rangle$. The arguments for these three properties follow.

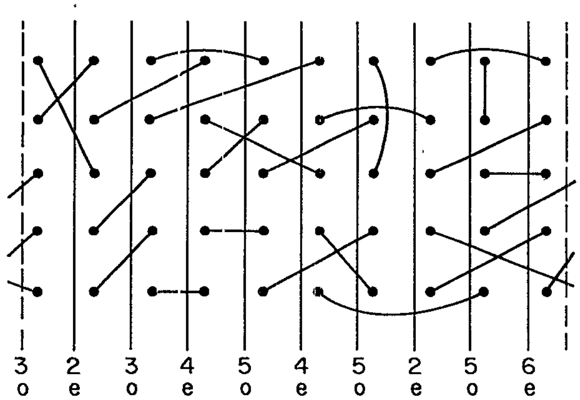

FIG. 2. A 5×10 square lattice with periodic boundary conditions exhibiting a short-range valence-bond configuration with on bond exceeding length 4. Solid vertical lines cut through each of the ten vertical gaps. Beneath each line is the number of bonds cut by that line and whether that number is even or odd. An alternating even-odd pattern will invariably arise when this is done on an odd width strip. When a valence-bond configuration is shifted through one lattice vector down the strip, the parity of the number of bonds sliced through a given vertical gap changes. Thus there are two classes of short-range valence-bond configurations: even odd and odd even. In the limit of infinite strip length, two bonds of differing types become orthogonal and the matrix element of any local Hamiltonian between them is zero. This implies that any translationally invariant state made up of short-range valence-bond states will be doubly degenerate in accordance with the Lieb-Schultz-Mattis theorem.

(i) $\langle\alpha|\beta\rangle=0$. The overlap of any two valence-bond states $|\alpha\rangle$ and $|\beta\rangle$ can be determined by constructing their transition graph and counting the number of loops that appear.[8,6,16] The overlap is then given by (up to an overall sign) $|\langle\alpha|\beta\rangle|=2^{N_{\alpha\beta}-ML/2}$, where $N_{\alpha\beta}$ is the number of loops and ML is the total number of lattice sites. As already stated, if $|\alpha\rangle\in(e,o)$ and $|\beta\rangle\in(o,e)$, the transition graph produced by $|\alpha\rangle$ and $|\beta\rangle$ invariably contains a nontrivial loop which encircles the torus at least once in the L direction. The number of bonds contained in such a loop must be more than L/l_0 so that the total number of loops in the transition graph cannot exceed $ML/2-L/l_0$ and $|\langle\alpha|\beta\rangle|<2^{-L/l_0}$. Thus, in the limit of infinite strip length, $|\alpha\rangle$ and $|\beta\rangle$ become orthogonal: $\langle\alpha|\beta\rangle=0$.

(ii) $\langle\alpha|H|\beta\rangle=0$. Next consider matrix elements of the form $\langle\alpha|H|\beta\rangle$ where H contains only short-range spin interactions. For simplicity consider only the nearest-neighbor Heisenberg Hamiltonian,

$$H=J\sum_{\langle i,j\rangle}\mathbf{S}_i\cdot\mathbf{S}_j\ ;$$

the result is more general. Then

$$\langle\alpha|H|\beta\rangle=J\sum_{\langle i,j\rangle}\langle\alpha|\mathbf{S}_i\cdot\mathbf{S}_j|\beta\rangle\ .$$

The state $(\mathbf{S}_i\cdot\mathbf{S}_j)|\beta\rangle$ is a singlet and can be written as a superposition of at most two normalized short-range valence-bond states. It follows from (i) that $\langle\alpha|\mathbf{S}_i\cdot\mathbf{S}_j|\beta\rangle\rightarrow\varnothing$ exponentially fast as $L\rightarrow\infty$ and so

$$\sum_{\langle i,j\rangle}\langle\alpha|\mathbf{S}_i\cdot\mathbf{S}_j|\beta\rangle\rightarrow\varnothing$$

as well. Thus the matrix element of H between any two valence-bond states $|\alpha\rangle\in(e,o)$ and $|\beta\rangle\in(o,e)$ is zero in the limit of infinite strip length: $\langle\alpha|H|\beta\rangle=0$.

(iii) Properties of $U_{2\mathrm{D}}$. When acting on a short-range valence-bond state it is easily shown that $U_{2\mathrm{D}}$ satisfies

$$U_{2\mathrm{D}}|\alpha\rangle=(-1)^{\gamma_\alpha}\prod_{\text{bonds}}[\cos(\Theta_b/2)+2i\sin(\Theta_b/2)S_b^z]|\alpha\rangle\ , \qquad (5)$$

where $\Theta_b=2\pi l_b/L$ (l_b is the length of the bond projected onto the x direction), S_b^z is the z component of the spin operator defined on the rightmost site associated with bond b, and γ_α equals the number of bonds which connect spins which are within l_0 to the left of column L with spins which are within l_0 to the right of column 1, i.e., bonds which are "cut" by the LSM slow-twist operator. The factor of -1 associated with each such bond occurs because only one of the two spins associated with the bond is twisted by $2\pi[+O(l_0/L)]$; it is this factor, $(-1)^{\gamma_\alpha}$, which allows $U_{2\mathrm{D}}$ to differentiate between (e,o) and (o,e) valence-bond states.

Since

$$\langle\alpha|S_{b_1}^zS_{b_2}^z\cdots S_{b_n}^z|\alpha\rangle=0\ ,$$

when $b_1\neq b_2\neq\cdots\neq b_n$, it follows that

$$\langle\alpha|U_{2\mathrm{D}}|\alpha\rangle=(-1)^{\gamma_\alpha}\prod_{\text{bonds}}\cos\Theta_b/2=(-1)^{\gamma_\alpha}[1-O(Ml_0^2/L)]\ . \qquad (6)$$

Because $U_{2\mathrm{D}}$ is a unitary operator, Eq. (6) implies that

$$U_{2\mathrm{D}}|\alpha\rangle=(-1)^{\gamma_\alpha}|\alpha\rangle+|\chi_\alpha\rangle\ , \qquad (7)$$

where

$$\langle\chi_\alpha|\chi_\alpha\rangle^{1/2}\sim O(Ml_0^2/L)\ .$$

Hence short-range valence-bond states become eigenstates of $U_{2\mathrm{D}}$ in the limit of infinite strip length, and the corresponding eigenvalue is $+1$ or -1 depending on whether the state belongs to (e,o) or (o,e).

Now consider a normalized superposition of short-range valence-bond states $\sum_\alpha C_\alpha|\alpha\rangle$. If each valence-bond state $|\alpha\rangle$ is of the (e,o) type, then the above-mentioned arguments tell us that

$$U_{2\mathrm{D}}\sum_\alpha C_\alpha|\alpha\rangle=\sum_\alpha C_\alpha|\alpha\rangle+\sum_\alpha C_\alpha|\chi_\alpha\rangle\ . \qquad (8)$$

Unfortunately, we have not succeeded in proving that

$$\left|\left|\sum_\alpha C_\alpha|\chi_\alpha\rangle\right|\right|\to 0$$

in the $L\to\infty$ limit. Each individual $|\chi_\alpha\rangle$ has a norm which goes to zero in this limit, but the number of short-range valence-bond states grows exponentially with L. One cannot prove that the $|\chi_\alpha\rangle$'s do not add up coherently in such a way that $\sum_\alpha C_\alpha|\chi_\alpha\rangle$ might have a finite norm in the $L\to\infty$ limit. We view our inability to exclude this possibility as a technical annoyance rather than a fundamental problem, and throughout the rest of this paper it will be assumed that (i)–(iii) hold for arbitrary superpositions of short-range valence-bond states.

Together (i)–(iii) imply that the topological decoupling is equivalent to the LSM theorem. If a Hamiltonian H defined on an odd-width strip has a translationally invariant ground state $|\psi_0\rangle$ which can be represented as a superposition of short-range valence-bond states, then a degenerate ground state can be constructed as follows. Express $|\psi_0\rangle$ as a superposition of its (e,o) and (o,e) components:

$$|\psi_0\rangle=(\tfrac{1}{2})^{1/2}(|\psi_\alpha\rangle+|\psi_\beta\rangle)\ ;$$

where $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$ are the projections of $|\psi_0\rangle$ into the (e,o) and (o,e) sectors, respectively. Then construct the state

$$|\psi_2\rangle=(\tfrac{1}{2})^{1/2}(|\psi_\alpha\rangle-|\psi_\beta\rangle)\ .$$

Properties (i) and (ii) then imply that $|\psi_2\rangle$ is orthogonal to $|\psi_0\rangle$ and becomes a degenerate ground state in the $L\to\infty$ limit. Because (iii) implies that

$$U_{2D}|\psi_\alpha\rangle=+|\psi_\alpha\rangle$$

and

$$U_{2D}|\psi_\beta\rangle=-|\psi_\beta\rangle\ ,$$

constructing $|\psi_1\rangle$ $(\equiv U_{2D}|\psi_0\rangle)$ will yield

$$(\tfrac{1}{2})^{1/2}(|\psi_\alpha\rangle-|\psi_\beta\rangle)\ .$$

Thus $|\psi_2\rangle$, the degenerate ground state constructed using the topological decoupling, and $|\psi_1\rangle$, the degenerate ground state constructed using the slow-twist operator, are the same state. In this sense the valence-bond decoupling on an odd-width strip can be viewed as a "rediscovery" of the LSM theorem.

IV. CONSEQUENCES FOR SPIN-LIQUID STATES

The LSM theorem and the (e,o), (o,e) decoupling both imply that a translationally invariant short-range RVB state, $|\psi_{\rm RVB}\rangle$, defined on an odd-width strip, must be at least twofold degenerate. The ground-state manifold will be spanned by $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$, the projection of $|\psi_{\rm RVB}\rangle$ onto the (e,o) and (o,e) valence-bond sectors, respectively. Thus any Hamiltonian which has $|\psi_{\rm RVB}\rangle$ as a ground state will also have $|\psi_\alpha\rangle$ as a ground state. Because $|\psi_\alpha\rangle$ is not an eigenstate of T, this would seem to imply a broken translational symmetry. This brings up a paradox: Short-range RVB states are explicitly constructed so as to have no broken symmetry, but the topological properties of valence-bond states on odd-width strips require a broken symmetry. The most natural resolution of this paradox is that this broken symmetry becomes unobservable in the limit of infinite (odd) strip width.

The broken translational symmetry inherent in short-range valence-bond states defined on odd-width strips can always be measured by an operator which acts on spins all the way across the strip. Such an operator must effectively determine the parity of the number of bonds in a given gap.[21] As the $M\to\infty$ limit is taken, this operator becomes *nonlocal*, i.e., it must act on an infinite $[\sim O(M)]$ number of spins. Presumably when M is finite, a local operator such as $\mathbf{S}_{X,Y}\cdot\mathbf{S}_{X+1,Y}$ will also be sensitive to this broken symmetry, but in such a way that the broken symmetry becomes unobservable in the $M\to\infty$ limit; e.g.,

$$\langle\psi_\alpha|\mathbf{S}_{X,Y_0}\cdot\mathbf{S}_{X+1,Y_0}|\psi_\alpha\rangle\sim C+(-1)^X D(M)\ , \qquad (9)$$

where C is independent of X and $D(M)\to 0$ as $M\to\infty$.

At first the idea of a broken translational symmetry which becomes unobservable in the thermodynamic limit may seem unnatural. However, there is at least one physical precedent: the fractional-quantum-hall effect in a finite system with periodic boundary conditions. Haldane[22] has shown that the energy eigenstates of such a system must be q-fold degenerate if the Landau-level filling factor is p/q. As for short-range RVB states on odd-width strips, this degeneracy involves states which can be transformed into one another by translation operators. The "broken translational symmetry" implied by this degeneracy is associated with the center-of-mass coordinate of the electrons. Thus a global operator acting on all the electrons in the system is required to measure it. If the ground state of the system is a fractional-quantum-hall fluid state, then this broken symmetry must become unobservable in the thermodynamic limit.

V. $L\times L$ LATTICES

If rather than a strip we consider an $L\times L$ square lattice (L even) with periodic boundary conditions, then short-range valence-bond states fall into *four* topologically distinct classes. Figure 3 shows representative valence-bond configurations from each class. Because L is even, an alternating even-odd pattern does not appear; instead lines cutting through either the vertical or horizontal gaps cut either all even or all odd numbers of bonds. The four sectors are then characterized by the parity of the bonds within the horizontal and vertical gaps. The arguments used in Sec. II to establish properties (i) and (ii) hold equally well here; in the $L\to\infty$ limit states belonging to different sectors become orthogonal and the matrix element of any local Hamiltonian between them is zero. A consequence of this decoupling is that any Hamiltonian with a ground state which may be written as a superposition of short-range valence-bond states will have four distinct ground states living in each of the

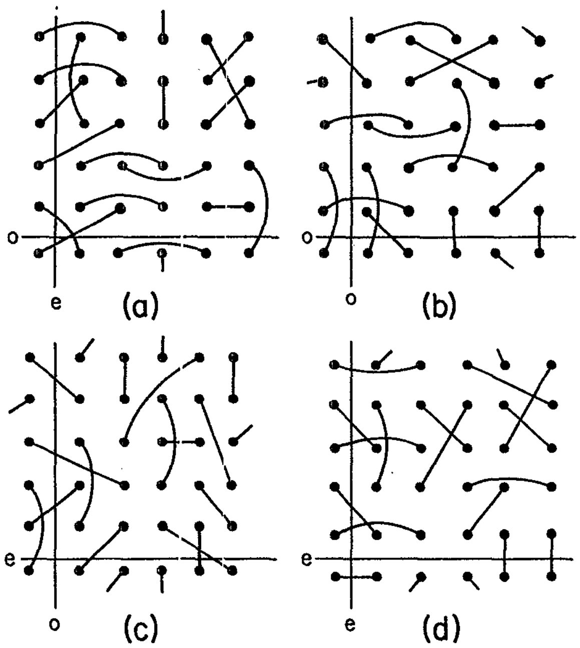

FIG. 3. Four short-range valence-bond configurations realized on 6×6 lattices with periodic boundary conditions. These states are characterized by the same slicing procedure shown in Fig. 1(a). Because the length and width of these configurations are both even, the parity of the number of bonds sliced in either the vertical or horizontal direction will all be the same. Thus short-range valence-bond states may be classified by these parities; i.e., configuration (a) depicts a valence-bond configuration with odd parity in the horizontal gaps and even parity in the vertical gaps. In the limit of infinite lattice size valence-bond states of differing types become orthogonal and the matrix element of any local Hamiltonian between them is zero. This suggests that the space of short-range valence-bond states decouples into *four* sectors in two dimensions.

four sectors shown in Fig. 3. In the following we denote the energy of these ground states by E_a, E_b, E_c, and E_d.

For the odd-width strip the translation operator T maps the (e,o) sector onto the (o,e) sector. Because T commutes with the Hamiltonian, this implies a twofold ground-state degeneracy if the ground state is a superposition of short-range valence-bond states. For the $L\times L$ lattice there is no analogous operator which maps states from one sector to another and which commutes with the Hamiltonian. Thus one cannot be sure of fourfold degeneracy for the $L\times L$ lattice, i.e., it need not be the case that $E_a=E_b=E_c=E_d$.[23] In fact, for a Hamiltonian having a spin-Peierls or valence-bond solid ground state, e.g., a state resembling that shown in Fig. 1(b), the ground states in the different topological sectors will contain line defects and thus not be degenerate with the purely crystalline ground state. Alternatively, if a Hamiltonian has a short-range RVB-type ground state, it seems likely that the ground-state energies in the different topological sectors will be equal in the $L\to\infty$ limit. Thus we expect that short-range RVB states on $L\times L$ lattices with periodic boundary conditions will be fourfold degenerate. Precisely this degeneracy has recently been noted in the nearest-neighbor RVB state by Read and Chakaborty.[24] Our observation should be viewed as a generalization of their work to RVB states with arbitrary (bounded) bond lengths.

Finally we note that Haldane[18] has recently argued that a non-Néel ground state of a half-integer spin frustrated antiferromagnet defined on a two-dimensional square lattice with periodic boundary conditions should be fourfold degenerate. The above-mentioned observations suggest that RVB states are consistent with this conjecture. As with the odd-width strip, this degeneracy is not associated with any locally observable broken symmetry and hence has no physical consequences. Of course crystalline states will also be fourfold degenerate because of their broken symmetry, and so will be consistent with Haldane's conjecture as well.[25]

VI. CONCLUSIONS

This paper has addressed the relationship between the LSM theorem and spin-liquid states constructed out of the short-range valence-bond basis. The main result is that the topological decoupling of the short-range valence-bond basis on odd-width strips appears to be equivalent to the LSM theorem. Both the decoupling and the LSM theorem imply the same twofold ground-state degeneracy. In one dimension the LSM theorem has important consequences for the nature of the spin-$\frac{1}{2}$ spin-liquid states. Because of the theorem, such states must have gapless excitations and power-law spin correlations. In two dimensions this is no longer the case, and the short-range RVB states introduced by Kivelson, Rokhsar, and Sethna[4] are completely consistent with the theorem, even though these states have a gap to spin excitations and exponentially decaying spin correlations. These states are consistent because the broken translational symmetry implied by the theorem on odd-width strips can become unobservable in the limit of infinite (odd) strip width. In addition, we have argued that the topological decoupling of short-range valence-bond states on $L\times L$ lattices with periodic boundary conditions implies short-range RVB states will be fourfold degenerate in the $L\to\infty$ limit, extending a recent observation to the case of arbitrary (bounded) bond lengths and in agreement with a conjecture by Haldane.

ACKNOWLEDGMENTS

I would like to acknowledge useful discussions with T.-L. Ho, S. A. Kivelson, and J. W. Wilkins. This work was supported by the U.S. Department of Energy (DOE), Basic Sciences, Division of Materials Research.

[1]L. Hulthén, Ark. Mat. Atron. Fys. **26a**, 1 (1938).

[2]P. W. Anderson, Science **235**, 1196 (1987).

[3]Example of locally observable broken symmetry: If a local operator, $O(x,y)$, exists with the property that $\langle\psi_0|O(x,y)|\psi_0\rangle$ depends on position, then $|\psi_0\rangle$ has a locally observable broken translational symmetry.

[4]S. A. Kivelson, D. S. Rokhsar, and J. P. Sethna, Phys. Rev. B **35**, 8865 (1987).

[5]P. W. Anderson, Mater. Res. Bull. **8**, 153 (1973).

[6]S. Liang, B. Doucot, and P. W. Anderson, Phys. Rev. Lett. **61**, 365 (1988).

[7]The results in this paper should be equally valid for valence-bond states in which the bond-length distribution is exponential.

[8]M. Kohmoto and Y. Shapir, Phys. Rev. B **37**, 9439 (1988).

[9]A Hamiltonian has been found (see Ref. 16) for which the ground state is a short-range RVB state. Although there is a gap to spin-1 excitations, there is a gapless collective mode with spin 0. There is some evidence that any frustrated spin-$\frac{1}{2}$ antiferromagnet for which this singlet mode is gapped will have a spin-Peierls ground state (see Ref. 25), and so there may in fact be no such thing as a short-range RVB state with a gap to *all* elementary excitations. We believe that this question is far from settled, and that the relevance of the LSM theorem to short-range RVB states is still of interest.

[10]E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. (N.Y.) **16**, 407 (1961).

[11]The LSM theorem applies to any translationally invariant Hamiltonian which contains only short-range spin interactions and which is rotationally invariant about any fixed axis. In this paper we consider only Hamiltonians which are fully rotationally invariant.

[12]I. Affleck, Phys. Rev. B **37**, 5186 (1988).

[13]I. Affleck and E. Lieb, Lett. Math. Phys. **12**, 57 (1986).

[14]S. A. Kivelson, D. S. Rokhsar, and J. P. Sethna, Europhys. Lett. **6**, 353 (1988).

[15]D. J. Thouless, Phys. Rev. B **36**, 7187 (1987).

[16]D. S. Rokhsar and S. A. Kivelson, Phys. Rev. Lett. **61**, 2376 (1988).

[17]A state $|\psi\rangle$ is doubly degenerate if it can be written as the sum of two states, $|\psi\rangle = A_\alpha|\psi_\alpha\rangle + A_\beta|\psi_\beta\rangle$, where $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$ are orthogonal and the matrix element of any local Hamiltonian between them is zero.

[18]F. D. M. Haldane, Phys. Rev. Lett. **61**, 1029 (1988).

[19]I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Commun. Math. Phys. **115**, 477 (1988).

[20]F. D. M. Haldane, Phys. Rev. B **25**, 4925 (1982).

[21]Such an operator could be a LSM slow-twist operator which spans the width of the strip but only acts on a finite length.

[22]F. D. M. Haldane, Phys. Rev. Lett. **55**, 2095 (1985).

[23]This is not quite true. An operator which rotates the system through 90° will map the sector labeled (a) in Fig. 3 to that labeled (c). Thus the ground states in these two sectors should be degenerate.

[24]N. Read and B. Chakaborty (unpublished).

[25]N. Read and S. Sachdev, Phys. Rev. Lett. **62**, 1694 (1989).