P H Y S I C A L   R E V I E W   L E T T E R S

Infinite-Randomness Fixed Points for Chains of Non-Abelian Quasiparticles

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One-dimensional chains of non-Abelian quasiparticles described by $SU(2)_k$ Chern-Simons-Witten theory can enter random singlet phases analogous to that of a random chain of ordinary spin-1/2 particles (corresponding to $k \rightarrow \infty$). For $k = 2$ this phase provides a random singlet description of the infinite-randomness fixed point of the critical transverse field Ising model. The entanglement entropy of a region of size L in these phases scales as $S_L \simeq \frac{\ln d}{3} \log_2 L$ for large L, where d is the quantum dimension of the particles.

A particularly exotic form of quantum order is possible in two space dimensions—so-called *non-Abelian* order [1]. In states with non-Abelian order, when certain localized quasiparticle excitations are present there is a low-energy Hilbert space whose dimensionality grows exponentially with the number of these quasiparticles. When these quasiparticles are well separated, this low-energy space becomes degenerate, and its states are characterized by purely topological quantum numbers, meaning they cannot be distinguished by local measurements. If these quasiparticles are then adiabatically moved around one another, unitary transformations corresponding to non-Abelian representations of the braid group are carried out on this degenerate space. Aside from their intrinsic scientific interest, recent attention has focused on the possibility of one day using non-Abelian states to perform fault-tolerant quantum computation [2,3].

Recently Feiguin *et al.* [4] have studied models of *interacting* non-Abelian quasiparticles, specifically uniform chains in which neighboring quasiparticles are close enough together to lift the degeneracy of the topological Hilbert space. In this Letter we study a related class of *random* interacting chains of non-Abelian quasiparticles. We are motivated both by [4] and by recent work of Refael and Moore [5,6] showing that the entanglement entropy of certain random one-dimensional models scales logarithmically with a universal coefficient. We find the same is true here for an infinite class of models.

Exact diagonalization studies [7–9] provide compelling evidence that the experimentally observed $\nu = 5/2$ fractional quantum Hall (FQH) state is a non-Abelian state described by the Moore-Read "Pfaffian" state [1]. This state belongs to a wider class of non-Abelian FQH states introduced by Read and Rezayi [10], labeled by index k. In this class, the $k = 1$ state is an ordinary (Abelian) Laughlin state, the $k = 2$ state is the Moore-Read state, and all subsequent integer k values describe new non-Abelian states. There is some numerical evidence [10,11] that the $k = 3$ Read-Rezayi state describes the experimentally observed $\nu = 12/5$ FQH state [12].

The quasiparticle excitations of the Read-Rezayi states with index k can be viewed (up to Abelian phases) as particle excitations in $SU(2)_k$ Chern-Simons-Witten theory [13]. These particles are characterized by their topological charge, a quantum number which can be viewed as a "q-deformed" spin [14]. At level k, topological charge can take the values $0, \frac{1}{2}, 1, \ldots, \frac{k}{2}$ and obeys the fusion rule,

$$s_1 \otimes s_2 = |s_1 - s_2| \oplus \cdots \oplus \min(s_1 + s_2, k - s_1 - s_2). \tag{1}$$

For $k \geq 2$ this implies $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$. Thus, when combining two particles with topological charge $1/2$ the resulting state can either have topological charge 0 or 1. For ordinary spin-1/2 particles the former would be referred to as a singlet and the latter as a triplet. We will use the same terminology for $SU(2)_k$ particles, though it should be noted that here there is no S_z degeneracy; i.e., there is only one "triplet" state. (For reviews of the general theory of non-Abelian particles, see [15,16].)

The total spin 0 sector of a one-dimensional chain of ordinary spin-1/2 particles is spanned by the set of all "noncrossing" singlet states, i.e., states in which pairs of particles form singlet bonds in such a way that these bonds do not cross [see Fig. 1(a)]. Furthermore, these noncrossing states are linearly independent [17], and their number, and hence the dimensionality of the spin 0 Hilbert space, grows asymptotically as 2^N for large N.

Using the generalized notion of singlet described above, noncrossing singlet states can also be used as a basis for the total topological charge 0 sector of a one-dimensional chain of $SU(2)_k$ particles [18]. In this case the interpretation is that any pair of particles connected by a singlet bond will fuse to topological charge 0 if brought together [19].

For N ordinary spin-1/2 particles, the overlap of two noncrossing singlet states $|\alpha\rangle$ and $|\beta\rangle$ can be computed by overlaying the two bond configuration and counting the number of closed loops, N_{loops}. The overlap is then given by $\langle \alpha | \beta \rangle = 2^{N_{\text{loops}} - N/2}$. For $SU(2)_k$ particles this overlap rule becomes $\langle \alpha | \beta \rangle = d^{N_{\text{loops}} - N/2}$, where $d = 2 \cos \frac{\pi}{k+2}$ is the "quantum dimension" of the particles [see Fig. 1(a)]
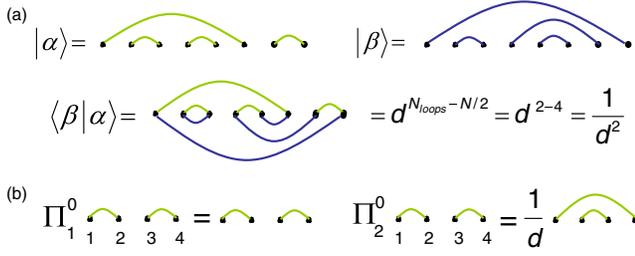
FIG. 1 (color online). (a) Two noncrossing singlet states for $SU(2)_k$ particles and their overlap. (b) Action of the singlet projection operators Π_1^0 (which acts on particles 1 and 2) and Π_2^0 (which acts on particles 2 and 3) on a particular noncrossing singlet state. The quantity d appearing in (a) and (b) is the quantum dimension of the particles.

[15,16]. For these values of d the noncrossing states are no longer linearly independent—they satisfy linear relations known as the Jones-Wenzl projectors [20] which reduce the size of the Hilbert space so that its dimensionality grows asymptotically not as 2^N, but as d^N for large N.

Consider a random one-dimensional chain of $SU(2)_k$ particles. Following [4], we assume that neighboring particles are close enough together so that the singlet and triplet fusion channels are split in energy, with the singlet lying lowest. The Hamiltonian describing this chain is then

$$H = -\sum_i J_i \Pi_i^0, \qquad (2)$$

where $J_i > 0$ is the energy splitting associated with particles at sites i and $i + 1$, and Π_i^0 is the singlet projection operator on these particles, the action of which on representative noncrossing singlet states is shown in Fig. 1(b). The uniform versions of these models ($J_i = J$) were studied numerically for $k = 3$ and analytically for all k in [4], where they were shown to be conformally invariant with central charge $c = 1 - 6/[(k + 1)(k + 2)]$.

Because the Hilbert space of this $SU(2)_k$ chain can be described using a noncrossing singlet basis, the usual real-space renormalization group (RG) approach based on decimating singlet bonds [21,22] can be straightforwardly applied to (2) when the J_i's are random. Each iteration of this procedure begins by finding the strongest bond in the chain, i.e., the J_i with the highest value, and making the approximation that the two particles connected by it fuse to topological charge 0 and so form a singlet bond.

The effective interaction $\tilde{J}$ between the two particles on either side of this singlet is then determined perturbatively as follows. Consider four neighboring particles and the associated three bond strengths J_1, J_2, and J_3, with $J_2 \gg J_1, J_3$ so that, as described above, a singlet forms between the two particles connected by J_2 (see Fig. 2). A straightforward generalization of the usual second-order perturbation theory calculation for ordinary spin-1/2 particles, but using the modified overlap rules shown in Fig. 1, then yields,

FIG. 2 (color online). One step in the decimation procedure.

$$\tilde{J} = (2/d^2)J_1 J_3 / J_2. \qquad (3)$$

Provided $d \geq \sqrt{2}$, which is the case for all $k \geq 2$ considered here, $\tilde{J}$ will always be less than the strength of the decimated bond J_2. Thus, as this procedure is iterated, high-energy bonds are systematically eliminated, leading eventually to a single noncrossing singlet state.

The RG flow produced by this decimation procedure can then be analyzed in the standard way [21,22]. Introducing the logarithmic bond strength variables $\beta_i = \ln(\Omega/J_i)$, where Ω is the largest remaining bond strength at any given stage of decimation, (3) can be written $\tilde{\beta} = \beta_1 + \beta_3 + \ln(2/d^2)$. Defining the flow parameter $\Gamma = \ln(\Omega_0/\Omega)$ where Ω_0 is the largest bond strength at the outset of the decimation procedure, and ignoring the $\ln(2/d^2)$ (i.e., taking $\tilde{\beta} \simeq \beta_1 + \beta_3$), an approximation which can be justified *a posteriori* due to the broad distribution of β's at the fixed point, an integro-differential equation, can be written down for the bond strength distribution $P_\Gamma(\beta)$ [21,22]. This distribution is defined so that when the flow parameter is Γ the fraction of bonds with logarithmic strength between β and $\beta + d\beta$ is $P_\Gamma(\beta)d\beta$. As shown by Fisher [22], for almost any initial random bond configuration, the bond strength distribution flows to the infinite-randomness fixed-point distribution, $P_\Gamma(\beta) = e^{-\beta/\Gamma}/\Gamma$. The resulting phase is known as a random singlet phase.

It follows that the random $SU(2)_k$ chains (2) flow to random singlet phases for all $k \geq 2$. In the limit $k \to \infty$ this phase corresponds to the usual random singlet phase for ordinary spin-1/2 particles [22]. For $k = 2$ we now show that the resulting phase can be mapped onto the infinite-randomness fixed point of the critical transverse field Ising model [23], thus providing a "random singlet" description of this fixed point.

We use the fact that $SU(2)_2$ particles can be represented using Majorana fermions operators γ_i [16]—operators which are self-conjugate ($\gamma_i^\dagger = \gamma_i$) and which satisfy the Clifford algebra $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$. Two Majorana fermions can be combined to form a usual fermion, so that, e.g., associated with neighboring sites i and $i + 1$ there is a fermion operator $c_{i,i+1}^\dagger = (\gamma_i + i\gamma_{i+1})/\sqrt{2}$ which satisfies the usual anticommutation relation $\{c_{i,i+1}, c_{i,i+1}^\dagger\} = 1$ and which anticommutes with any similar fermion operator constructed out of a different pair of Majorana fermions. The Fermi mode associated with this pair can then be occupied (corresponding to topological charge 1) or unoccupied (corresponding to topological charge 0). The

singlet projection operator is then $\Pi_i^0 = 1 - c_{i,i+1}^\dagger c_{i,i+1}$, which in the Majorana representation is $\Pi_i^0 = i\gamma_i\gamma_{i+1}$.

To map the $SU(2)_2$ chain onto the transverse field Ising model we first group together neighboring pairs of Majorana fermions. Letting the index j label these pairs, each of which consists of a right Majorana fermion (γ_j^R) and a left Majorana fermion (γ_j^L), the Hamiltonian (2) can be written

$$H = -\sum_j h_j i\gamma_j^L\gamma_j^R - \sum_j J_j i\gamma_j^R\gamma_{j+1}^L. \qquad (4)$$

Here h_j corresponds to the coupling within the jth pair, and J_j corresponds to the coupling between the rightmost particle in the jth pair and the leftmost particle in the $(j + 1)^{\text{st}}$ pair. The usual Jordan-Wigner transformation (see, for example, [24]), $\gamma_j^L = \sigma_j^x\prod_{k=1}^{j-1}\sigma_k^z$ and $\gamma_j^R = \sigma_j^y\prod_{k=1}^{j-1}\sigma_k^z$, then maps (4) onto the random transverse field Ising model,

$$H = \sum_j h_j\sigma_j^z + \sum_j J_j\sigma_j^x\sigma_{j+1}^x. \qquad (5)$$

Because h_j and J_j are drawn from the same distribution, the model is at its critical point.

The usual decimation procedure for the transverse field Ising model involves two separate steps—either forming ever larger "superspins" when the strongest interaction is an Ising interaction (J) or decimating these superspins when the strongest interaction is a magnetic field strength (h) [23]. The $SU(2)_2$ "random singlet" view of this decimation provides a unified description of these two steps. Figure 3 shows a random singlet state produced by decimation and a reference "dimer" state in which bonds connect pairs of Majorana fermions which correspond to single spins in the transverse field Ising model. Overlaying these two states produces closed loops which, in the transverse field Ising model, correspond to decimated superspins. Essentially, as the decimation which produces the random singlet state shown in the figure proceeds, any time a bond forms which does not close a loop this corresponds

to eliminating an Ising interaction and increasing the number of spins contributing to a superspin. Then, when a bond forms which closes a loop, the corresponding superspin is frozen along the direction of the applied field and decimated.

Recently Refael and Moore [5] have shown that the entanglement entropy associated with the infinite-randomness fixed points of both the random spin-1/2 Heisenberg chain ($k \to \infty$) and the transverse field Ising model ($k = 2$) have universal scaling properties which can be used to generalize the notion of central charge to one-dimensional quantum critical systems which are not conformally invariant. We now show that the same is true for all the $SU(2)_k$ infinite-randomness fixed points. The entanglement entropy of these states is calculated by treating a contiguous segment of the chain consisting of L particles as a subsystem (denoted A) of the full chain. Tracing out the degrees of freedom of the rest of the chain then yields a reduced density matrix ρ_A. The entanglement entropy is the average over realizations of disorder of the von Neumann entropy of this reduced density matrix, $S_L = -\langle\text{Tr}\rho_A\log_2\rho_A\rangle$.

In random singlet states the calculation of S_L for large L can be done, as in [5], by counting the number of singlet bonds which connect sites in region A with sites outside of it, averaging over realizations of disorder, and then multiplying the result by the entanglement entropy associated with each bond. All the $SU(2)_k$ random singlet states discussed here are governed by the same fixed-point bond distribution as that considered in [5], so the result of that work that the average number of bonds contributing to the entanglement scales as $\frac{1}{3}\ln L$ for large L holds here as well.

To compute the entanglement entropy per bond for $SU(2)_k$ particles, imagine forming N singlet pairs, with one particle from each pair taken to be in subsystem A and the other in subsystem B, as shown in Fig. 4. This figure also shows a Schmidt decomposition of this state using a basis in which ovals are drawn around particles in topo-

FIG. 3 (color online).   "Random singlet" view of a decimated random transverse field Ising model. A random singlet state (green) is overlaid with a dimer state (blue). In the dimer state bonds connect pairs of $SU(2)_2$ particles which are mapped onto the spin-1/2 degrees of freedom of the transverse field Ising model (bottom row of dots). Closed loops then correspond to decimated superspins, indicated by solid arrows.

FIG. 4 (color online).   Schmidt decomposition of a state of N pairs of $SU(2)_k$ particles connected by singlet bonds. The states in the decomposition are expressed using a basis in which circles enclose particles in topological charge eigenstates. The sum is over all $s_2, s_3, \ldots, s_N$, consistent with the fusion rule (1).

logical charge eigenstates. The Schmidt coefficients (λ_{s_N} in Fig. 4) can be obtained using standard calculation techniques for non-Abelian particles [15,16]. They depend only on the total topological charge s_N of the particles in region A (or equivalently region B) of the corresponding state in the Schmidt decomposition, and they are given by $\lambda_{s_N} = [2s_N + 1]/d^N$, where we have introduced the q integers $[m] = (q^{m/2} - q^{-m/2})/(q^{1/2} - q^{-1/2})$ with $q = \exp[i2\pi/(k + 2)]$.

The von Neumann entropy of the reduced density matrix ρ_A obtained by tracing out the degrees of freedom in region B is then [25], $S_A = -\sum_{s_N} D(N, s_N)\lambda_{s_N}\log_2\lambda_{s_N}$, where $D(N, s_N)$ is the dimensionality of the space of N $SU(2)_k$ particles with total topological charge s_N. Using the fact that, for large N, $D(N, s_n) \simeq [2s_N + 1]d^N/\mathcal{D}^2$ where $\mathcal{D}^2 = \sum_{s=0}^{k/2}[2s + 1]^2$ [15,16], it follows that $S_A \simeq N\log_2 d - O(\log_2 k)$ for $N \gg k$. Thus for large N the entanglement per bond is $\log_2 d$, reflecting the fact that the size of the Hilbert space of N particles grows asymptotically as d^N [26].

Returning to the $SU(2)_k$ random singlet phases, multiplying the average number of bonds leaving a region of size L ($\simeq \frac{1}{3} \ln L$) by the entanglement per bond ($\simeq \log_2 d$) yields

$$S_L \simeq (\ln d/3)\log_2 L. \qquad (6)$$

Following [5], if we compare (6) with the entanglement entropy of conformally invariant one-dimensional systems, $S_L \simeq \frac{c}{3}\log_2 L$ where c is the central charge [24,27–29], it is natural to define an "effective central charge" of $\tilde{c} = \ln d$ for these phases. In the $k \rightarrow \infty$ limit, corresponding to the ordinary $SU(2)$ random singlet phase with $d = 2$, we have $\tilde{c} = \ln 2$, and for $k = 2$, corresponding, as shown above, to the critical point of the random transverse field Ising model with $d = \sqrt{2}$, we have $\tilde{c} = \frac{1}{2} \ln 2$, both of which agree with results obtained in [5].

Finally, we note that for the $SU(2)_k$ chains considered here the effective central charge of the disordered model, $\tilde{c} = \ln d$ is always less than the central charge of the uniform model, $c = 1 - 6/[(k + 1)(k + 2)]$ [4], though the simple relation $\tilde{c} = \ln 2 \times c$ emphasized in [5] only holds for $k \rightarrow \infty$ and $k = 2$. This is consistent with the generalized "c theorem" envisioned in [5] which supposes that the effective central charge decreases along RG flows between quantum critical points. However, it should be emphasized that this "theorem" is not a rigorous result. In particular, Santachiara [30] has shown that it is violated by RG flows from the uniform to disordered phases of the Z_n parafermionic Potts model for $n \geq 42$.

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