Numerical Analysis of Quasiholes of the Moore-Read Wave Function

M. Baraban,[1] G. Zikos,[2] N. Bonesteel,[2] and S. H. Simon[3]

[1]*Department of Physics, Yale University, 217 Prospect Street, New Haven, Connecticut 06511, USA*

[2]*Department of Physics and National High Magnetic Field Laboratory, Florida State University, Tallahassee, Florida 32310, USA*

[3]*Rudolf Peierls Centre for Theoretical Physics, Oxford University, 1 Keble Road, Oxford OX1 3NP, United Kingdom*

(Received 26 January 2009; published 11 August 2009)

We demonstrate numerically that non-Abelian quasihole (qh) excitations of the $\nu = 5/2$ fractional quantum Hall state have some of the key properties necessary to support quantum computation. We find that as the qh spacing is increased, the unitary transformation which describes winding two qh's around each other converges exponentially to its asymptotic limit and that the two orthogonal wave functions describing a system with four qh's become exponentially degenerate. We calculate the length scales for these two decays to be $\xi_U \approx 2.7\ell_0$ and $\xi_E \approx 2.3\ell_0$, respectively. Additionally, we determine which fusion channel is lower in energy when two qh's are brought close together.

DOI: 10.1103/PhysRevLett.103.076801 PACS numbers: 73.43.−f

The proposal to use quantum Hall states as a platform for quantum computation has spurred a great deal of interest [1–3]. These quantum Hall systems are believed to have natural "topological" immunity to decoherence and therefore hold particular promise for quantum computation. In so-called non-Abelian quantum Hall systems, the ground state is highly degenerate in the presence of quasiparticles (qp's), and this degenerate space can be used to store quantum information. Operations on this space are then performed by adiabatically dragging qp's around each other, thus "braiding" their world lines in $2 + 1$ dimensions.

Although there is currently no definitive experimental evidence that non-Abelian quantum Hall states even exist, the community now strongly suspects [1] that the quantum Hall plateau observed at Landau level (LL) filling fraction $\nu = 5/2$ is the non-Abelian Moore-Read (MR) phase [4] (or its closely related particle-hole conjugate [5]). While the MR phase is, strictly speaking, not capable of universal topological quantum computation (computation by braiding qp's around each other at large distances), a scheme has been devised [6] that in principle allows error-free quantum computation by supplementing these topological processes with nontopological processes where qp's are moved together and allowed to interact. Furthermore, the MR phase is frequently viewed as the simplest paradigm of a non-Abelian state of matter, and is therefore a logical starting point for detailed analysis [1].

In order for topological (or partially topological) schemes for quantum computation to be scalable (i.e., to allow large scale quantum computation), a number of crucial conditions must hold [1]. Condition (1) As all of the qp's are moved apart from one another, the splitting of the energy levels of the putatively degenerate ground state space must converge to zero at least as fast as e^{-R/ξ_E} where R is the minimum distance between qp's. In the literature, there has been numerical work suggesting that condition (1) may not be true [7] for the MR state. One of the goals of our work is to perform more precise numerical calculations to determine whether this numerical conclusion holds up to more careful scrutiny. Condition (2) As qp's are moved apart from each other, the unitary transformation that results from adiabatically dragging one qp around another must converge to its asymptotic limit at least as fast as e^{-R/ξ_U}. For the MR state, several theoretical arguments suggest that this is true [8–10]; however, in these theoretical works, the precise length scale ξ_U remains unknown. Presumably, ξ_E and ξ_U are both on the scale of a magnetic length multiplied by some number of order unity. If this number of "order unity" happens to be very large, it could in principle start to cause trouble for practical implementation of topological schemes. We will explicitly determine both ξ_U and ξ_E numerically. Finally, Condition (3) One must be able to measure the topological quantum number associated with a group of qp's. Proposals have been made that such quantum numbers can be measured using interferometry [1,2,11]. However, this scheme has turned out to be very difficult experimentally. Another possible way to measure the topological quantum number of, say, two qp's, is to move the qp's microscopically close and precisely measure the force between them (or equivalently the energy change of moving them). While this may not sound any easier, it nonetheless proposes a different route to making this measurement should interferometry prove to be impossible. In the current Letter, we will attempt to numerically evaluate this energy change and show how it reflects the quantum number of a pair of qp's. See Ref. [12] for a similar analysis of the Kitaev model.

Our numerical work is performed on a spherical geometry with a monopole of flux N_ϕ at the center of the sphere and N electrons on the surface. For the MR state [4], N_ϕ is given by $N_\phi = 2N - 3 + n_{\rm qh}/2$ and $n_{\rm qh}$ is the number of quasiholes (qp's with positive charge). The radius of the sphere is $(N_\phi/2)^{1/2}\ell_0$ where ℓ_0 is the magnetic length. For the purpose of stating the decay lengths ξ_U and ξ_E, the distance R between quasiholes (qh's) will be written in terms of the chord length. The definitions of ξ_E and ξ_U are given below. It should be noted that while their precise values depend on the particular qh configurations used in our calculations, alternate definitions for different qh configurations will give results that only differ by factors of order unity.

We consider the MR wave function (WF) in the presence of qh's which is defined as the zero energy space of a special short-ranged three-body interaction [13]. Although this is just a model interaction, the ground state WF turns out to be an accurate approximation for more realistic interactions [14]. Thus, our calculations are variational in nature. Pairs of qh excitations carry the topological quantum number "1" or "ψ" which represents the two states of a qubit and the degeneracy [15] of a system with n_{qh} qh's is $2^{(n_{\mathrm{qh}}/2)-1}$.

We start by considering the case of two qh's for which the ground state is unique. In this case, we can address condition (2) above by calculating the braiding statistics of these two qh's. To do so in the spherical geometry, we compute the Berry phase accumulated when one qh is moved adiabatically around the equator of the sphere while the second qh is held fixed first on the north pole and then on the south pole. Both these Berry phases have contributions from the statistical phase associated with the two qh's, and the Aharanov-Bohm phase due to the applied magnetic field. To isolate the statistical phase, we therefore compute the *difference* between these two phases. In the planar geometry, this difference would correspond to the change in the Berry phase when one qh is moved in a closed loop while a second qh is held fixed first inside the loop and then outside the loop.

The Berry phases are all calculated numerically using a Monte Carlo method essentially identical to that described in Ref. [16]. We use the MR WF with two qh's, which is not an exact WF for the realistic Coulomb interaction, but is quite accurate nonetheless as prior numerical work has demonstrated [14]. When we drag qh's, we can think of having added a highly localized potential well to the system whose position moves as a function of time. However, since our Berry phase calculation does not involve a detailed Hamiltonian *per se*, our results are independent of the form of this potential well. (Further details of the methods used will be given in Ref. [17]). For the cases of either an even or odd number of electrons on the sphere, the two qh's together must have topological quantum numbers 1 or ψ, respectively. The statistical phase is then expected [1,4,8,10] to converge either to zero (if the quantum number is 1) or π (if the quantum number is ψ) as the distance between the qh's is increased. Indeed, for an even number of electrons, we show in Fig. 1 that as the sphere is made larger, the convergence is exponential and the decay scale is roughly $\xi_U \approx 2.7\ell_0$. Similar results were

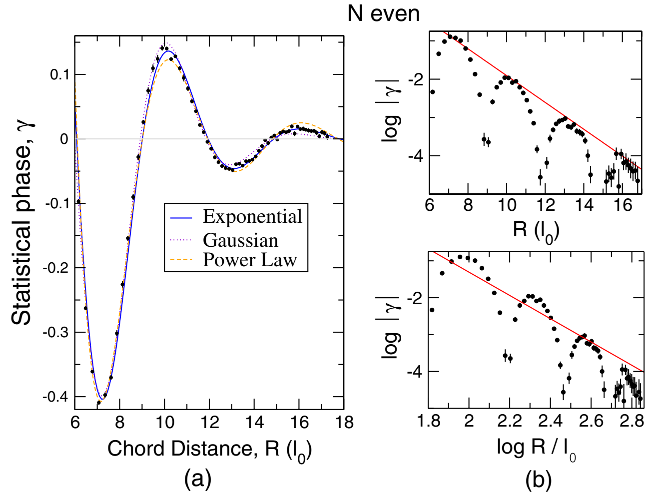

FIG. 1 (color online). Statistical phase for winding one qh around another in the spherical geometry as defined in the text. Data are shown for the case of even total numbers of electrons for which the topological quantum number of the pair of qh's is 1. In (a), we have plotted the statistical phase versus the chord distance R between the two qh's and fit the data using $\cos(a\frac{R}{l_0} + b)$ for the oscillatory part and a decay term that is either exponential, Gaussian or power law. We fit the data starting at $R = 6.48 l_0$ and find the value of the reduced χ^2 is smallest for exponential decay with a value of 1.42 while for power law and Gaussian decays, it is 7.22 and 5.52, respectively. The good fit to exponential decay is also confirmed when we plot the absolute value of the data on log and log-log scales (b) and perform linear fits to the extrema of the oscillations. The linear fit is clearly much better on the log plot demonstrating that the oscillations decay exponentially rather than as a power law.

obtained for the case of an odd number of electrons where the phase converges exponentially to π with roughly the same decay scale.

The difference between the even and odd case can be interpreted as the non-Abelian component—i.e, the part of the phase that depends on which topological sector the two qp's are in. We conclude that this non-Abelian contribution does indeed converge exponentially with increasing system size as desired by condition (2). (Ideally we would like to determine the unitary transformation that occurs on this two dimensional ground state space when particles are braided around each other as in Ref. [16]. However, we have found that it is currently numerically too demanding to demonstrate exponential convergence in this more complicated situation).

The oscillations in Fig. 1 (and in the later Figures) are not unexpected. In the closely related system of a p-wave paired superfluid, the oscillating form of the wave functions can be calculated explicitly [18,19]. However, in this quantum Hall system, those results would only be qualitative.

To address condition (1) above, we now turn to the case of four qh's and restrict ourselves to an even number of electrons. We implement a trial WF approach using the MR WF with qh's, which is the ground state of a special three-body interaction, but we will evaluate its energy with a first excited LL Coulomb interaction [20], for which the MR WF is not the exact ground state. We nonetheless expect this hybrid approach to give accurate results because the MR WF is an extremely accurate approximation of the exact ground state of the Coulomb interaction. Our calculation is also an exact statement about the lowest order perturbation of the special three-body interaction towards the Coulomb point [14].

For the MR WF with four qh's, there are two putatively degenerate ground state wave functions [4,8]. Using a spherical geometry, we place four qh's on the corners of an equilateral tetrahedron and implement standard Monte Carlo procedures to evaluate the energy splitting between the two eigenstates of the interaction within the two dimensional ground state subspace (Details will be presented in Ref. [17]). Results are presented in Fig. 2 as a function of system size, and indeed it appears that the two blocks become degenerate exponentially as the distance between qp's increases as required by condition (2) with a decay length of $\xi_E \approx 2.3\ell_0$. This result appears to contradict results of Ref. [7] which claimed an algebraic rather than exponential decay. Although the methods used by the two works are essentially identical, in Ref. [7] the MR state is studied in the lowest LL (where MR is known not to be a good trial WF) whereas we have studied it in the first LL where it is known to be a very good trial WF and thought to be experimentally relevant. Differences with Ref. [7] could occur because of differing levels of Monte Carlo error as well. The numerical difficulty of collecting data is substantial, so admittedly our error bars are currently somewhat larger than desirable. However, we will continue to collect data, and these results will almost certainly improve.

Finally, we turn to the issue of measurement, condition (3). Here, we start with four qh's at the corners of a tetrahedron on a relatively large sphere ($N = 40$) where the two ground state wave functions are close to degenerate. We then move qh 2 close to 1 and observe the change in energy of the two wave functions. It turns out (and we will show in detail in Ref. [17]) that if we choose to move the qh's together along an appropriately chosen path, then the conformal block wave functions defined in Ref. [8] diagonalize the interaction. These two conformal block wave functions, known as $|1\rangle$ and $|\psi\rangle$, are constructed such that the pair of qh's 1 and 2 have topological quantum number 1 and ψ, respectively. The results of such a calculation are shown in Fig. 3. We see that the energy of moving two qh's together is always positive simply due to the Coulomb repulsion. However, the energy is substantially greater when the two qh's are in the $|1\rangle$ state compared to the $|\psi\rangle$ state. To our knowledge, this result was not predicted and may be attributed to the fact that the electron density vanishes in the $|1\rangle$ state when two qh's approach each other, but remains nonzero in the $|\psi\rangle$ state resulting in a more extended object [17]. Thus our calculation makes the first mapping between a proposed measurement of the

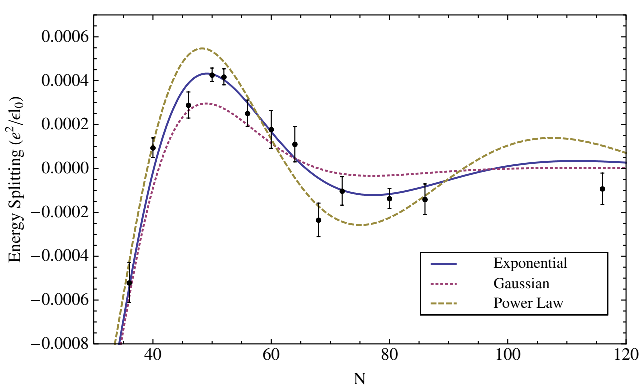

FIG. 2 (color online). Energy splitting of the eigenstates on a sphere with four qh's as a function of system size. The four qh's are placed on the corners of an equilateral tetrahedron. The data is fit with the function $\cos(a\sqrt{N} + b)$ multiplied by an exponential, Gaussian, or power law decay function. The distance between particles grows as $\sqrt{N}$, so this is essentially the same fit as used in Fig. 1. The data is fit starting with $N = 12$ (not shown because the fits are nearly identical at the small N values). The reduced χ^2 values for the exponential, Gaussian, and power law fits are 3.39, 7.73, and 6.49, respectively, which helps confirm our expectation that two energies become exponentially degenerate as the qh's move apart. We have not shown the log and log-log plots here because such plots discard the sign, and in the absence of a higher density of points, are hard to interpret.

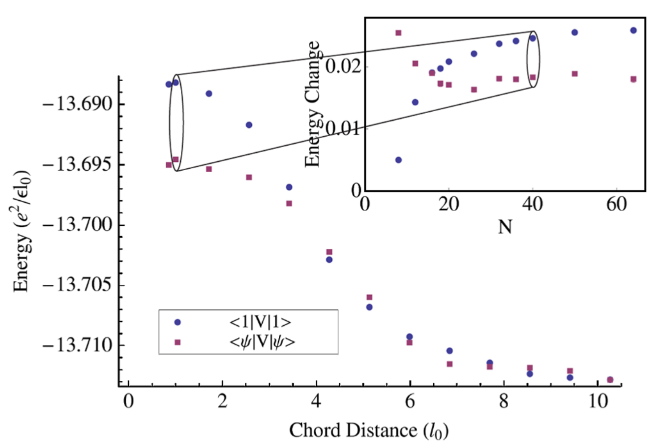

FIG. 3 (color online). This figure shows the total energy of the $N = 40$ system with four qh's as qh 2 is moved closer to its pair, qh 1. qh 1 is located at the north pole, and qh 2 is moved along a path that keeps a certain analytic form [8] for the trial states $|1\rangle$ and $|\psi\rangle$ precisely orthogonal [17]. The two states of the system, $|1\rangle$ and $|\psi\rangle$, which are nearly degenerate when the qh's are well separated, split as the qh's approach each other. The inset shows the energy needed to move qh 2 within ℓ_0 of qh 1 for different values of N. We find that it takes more energy to bring the two qh's together when the system is in state $|1\rangle$ than in state $|\psi\rangle$, and that the energy splitting is on the order of $0.01e^2/\epsilon\ell_0$.

energy of two qh's and what this would indicate in terms of determining their topological quantum number. (Obviously if one were moving a qp together with a qh, the $|1\rangle$ state would have lower energy). We also point out that this result may be significant in regards to the possibility of qh's condensing into daughter states as proposed in Refs. [21,22].

The magnitude of the energy splitting of the two states, when two qh's are very close to each other (within ℓ_0), is measured to be roughly $0.01 e^2/\epsilon\ell_0$ which in a real system corresponds to roughly 1 K, a rather small energy to be measured. To make matters worse, this measured energy should be considered to be an upper bound, as mixing with states above the gap will be substantial and could easily reduce this energy scale (the experimentally measured gap itself is less than 1 K in the very best samples, although theoretically without disorder the gap could be almost 2.5 K. See Ref. [1] and therein). Nonetheless, this numerical work gives the 1st order of magnitude estimate for how large the splitting due to topological quantum numbers is likely to be compared to the overall Coulomb energy between the two qh's.

The decay length scales and energy scales that we calculate above are also extremely relevant to Majorana tunneling. Plugging in real numbers, we find that at a separation of about 0.1 μm, the energy difference will be about 80 mK with the sign of the tunneling amplitude depending sensitively on the distance. These scales have implication to interferometry experiments [23] where tunneling occurs between edge and bulk as well as for a Majorana hopping problem where tunneling occurs between many bulk qh's [22].

To summarize, we have used Monte Carlo techniques to examine several key properties of the MR WF with qh's. Note that because our calculations do not incorporate LL mixing terms (which are expected to be small), they are equally applicable to the recently proposed AntiPfaffian WF [5]. We find that both the unitary transformation associated with adiabatic transport and the energy splitting of putatively degenerate states converge exponentially with increasing distance between qp's, and we explicitly extract the decay lengths. Encouragingly, the decay lengths are on the order of a magnetic length which suggests that qp spacing should not be a barrier to physical implementation of topological operations. Further, we examine the energy splitting that occurs when two qh's are moved together. We find that the $|1\rangle$ state of these two particles is of *higher* energy and we measure this energy splitting between $|1\rangle$ and $|\psi\rangle$. Although this energy splitting is small, it gives experimentalists another way to measure topological quantum numbers in these systems. Many more details of this work will be presented in an upcoming publication [17].

We thank N. Read, S. M. Girvin, P. Bonderson, and J. K. Slingerland for helpful input. M. B.'s computer time was supported by NSF Grants No. DMR-0603369 and No. DMR-0653377. G. Z. and N. E. B. are supported by US DOE Grant No. DE-FG02-97ER45639.

[1] See, for example, C. Nayak *et al.*, Rev. Mod. Phys. **80**, 1083 (2008); arXiv:0707.1889, and references therein.
[2] S. DasSarma, M. Freedman, and C. Nayak, Phys. Rev. Lett. **94**, 166802 (2005).
[3] P. Bonderson, K. Shtengel, and J. K. Slingerland, Phys. Rev. Lett. **97**, 016401 (2006); A. Stern and B. I. Halperin, Phys. Rev. Lett. **96**, 016802 (2006); M. Freedman *et al.*, Ann. Phys. (N.Y.) **310**, 428 (2004).
[4] G. Moore and N. Read, Nucl. Phys. **B360**, 362 (1991).
[5] M. Levin, B. I. Halperin, and B. Rosenow, Phys. Rev. Lett. **99**, 236806 (2007); S.-S. Lee *et al.*, Phys. Rev. Lett. **99**, 236807 (2007); Note that the results of the current Letter can easily be translated to apply to qp's of the AntiPfaffian WF.
[6] S. Bravyi, Phys. Rev. A **73**, 042313 (2006).
[7] C. Toke, N. Regnault, and J. K. Jain, Solid State Commun. **144**, 504 (2007).
[8] C. Nayak and F. Wilczek, Nucl. Phys. **B479**, 529 (1996).
[9] V. Gurarie and C. Nayak, Nucl. Phys. **B506**, 685 (1997).
[10] N. Read, Phys. Rev. B **79**, 045308 (2009).
[11] B. Overbosch and F. Bais, Phys. Rev. A **64**, 062107 (2001).
[12] V. Lahtinen *et al.*, Ann. Phys. (N.Y.) **323**, 2286 (2008).
[13] M. Greiter, X. G. Wen, and F. Wilczek, Nucl. Phys. **B374**, 567 (1992).
[14] R. H. Morf, Phys. Rev. Lett. **80**, 1505 (1998); E. H. Rezayi and F. D. M. Haldane, Phys. Rev. Lett. **84**, 4685 (2000); See also references within Ref. [1].
[15] A system with uncertain parity of N would have degeneracy $2^{n_{\mathrm{qh}}/2}$. See Ref. [1].
[16] Y. Tserkovnyak and S. H. Simon, Phys. Rev. Lett. **90**, 016802 (2003).
[17] G. Zikos, M. Baraban, N. Bonesteel, and S. H. Simon (to be published).
[18] V. Gurarie and L. Radzihovsky, Ann. Phys. (N.Y.) **322**, 2 (2007).
[19] M. Cheng *et al.*, arXiv:0905.0035.
[20] K. Park *et al.*, Phys. Rev. B **58**, R10167 (1998).
[21] P. Bonderson and J. K. Slingerland, Phys. Rev. B **78**, 125323 (2008).
[22] N. Read and A. W. W. Ludwig, Phys. Rev. B **63**, 024404 (2000); A. Kitaev (unpublished); A. Feiguin *et al.*, Phys. Rev. Lett. **98**, 160409 (2007); C. Gils *et al.*, Phys. Rev. Lett. **103**, 070401 (2009); E. Grosfeld and A. Stern, Phys. Rev. B **73**, 201303 (2006).
[23] B. Rosenow *et al.*, arXiv:0906.0310; W. Bishara and C. Nayak, arXiv:0906.0327.