Monte Carlo Comparison of Quasielectron Wave Functions

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Abstract

Variational Monte Carlo calculations of the quasielectron and quasihole excitation energies in the fractional quantum Hall effect have been carried out at filling fractions $\nu = 1/3$, $1/5$, and $1/7$. For the quasielectron both the trial wave function originally proposed by Laughlin and the composite fermion wave function proposed by Jain have been used. We find that for long-range Coulomb interactions the results obtained using these two wave functions are essentially the same, though the energy gap obtained using the composite fermion quasielectron is slightly smaller, and closer to extrapolated exact-diagonalization results.

I. INTRODUCTION

Shortly after the discovery of the fractional quantum Hall effect (fractional QHE) [1] Laughlin [2] introduced a set of trial wave functions describing the 'parent' quantum Hall states occurring at Landau level filling fraction $\nu = 1/q$ where q is an odd integer. In addition to the ground state, Laughlin introduced trial wave functions describing fractionally charged quasielectron (e/q) and quasihole ($-e/q$) excitations. From the very beginning it was clear that the wave function for the quasihole, with its simple Jastrow form, was more natural than the wave function for the quasielectron, which contains explicit derivatives with respect to electron coordinates. This difference is reflected, for example, in the fact that while there exists a Hamiltonian for which the Laughlin ground state and quasihole wave functions are exact (zero energy) eigenstates [3] there exists no such simple Hamiltonian for which the quasielectron wave function is also an exact eigenstate.

According to the composite fermion theory, proposed by Jain [4], the fractional QHE corresponds to an *integer* QHE of composite fermions – electrons bound to an even number of statistical flux quanta. This identification leads to a procedure for constructing fractional QHE trial states by first constructing integer QHE states, then multiplying by a Jastrow factor which binds an even number of vortices to each electron, and finally projecting the resulting state onto the lowest Landau level. The wave function obtained using this procedure for the parent quantum Hall state is identical to Laughlin's ground state, and the same is true for the quasihole wave function. However, the composite fermion quasielectron wave function is not identical to the one proposed by Laughlin.

To date, the best estimate of the energy gap for creating a quasielectron-quasihole pair with infinite separation at $\nu = 1/3$ computed using Laughlin's trial states was obtained by Morf and Halperin, using the disk [5] and spherical [6] geometries, with the result $\Delta_L \simeq 0.092 \pm 0.004 e^2/\epsilon l_0$ where $l_0 = \sqrt{\hbar c/eB}$ is the magnetic length. This may be compared with the result of Bonesteel [7] using the composite fermion quasielectron wave function of $\Delta_{CF} \simeq 0.106 \pm 0.003 e^2/\epsilon l_0$. It therefore appears that the energy gap computed using the

Laughlin quasielectron is over %10 lower than that obtained using the composite fermion theory, and so one might conclude that the Laughlin quasielectron wave function provides a (slightly) better description of the physical quasielectron state. More recently, Girlich and Hellmund [8] have shown that for the truncated (V_1) pseudopotential interaction introduced by Haldane [3], the interaction for which the Laughlin ground state and quasihole state are exact eigenstates, the Laughlin quasielectron has an energy which is %18 *higher* than the composite fermion quasielectron. These authors go on to speculate that the same would be true for the Coulomb interaction, and this has motivated us to reexamine the calculations of Morf and Halperin.

Taking advantage of the availability of significantly faster computers, and performing a better extrapolation of finite size results to the thermodynamic limit, we find that the $\nu = 1/3$ energy gap computed using the Laughlin quasielectron extrapolates to $\Delta \simeq 0.110(2)e^2/\epsilon l_0$, a significantly higher result than previously obtained. This result is consistent with Girlich and Hellmund [8], though we find that for the Coulomb interaction the difference between the two energy gaps is quite small (less than % 5) indicting that both wave functions provide adequate descriptions of the true quasielectron. We have also performed what we believe are the first calculations of the energy gap using Laughlin's quasielectron wave functions for filling fractions $\nu = 1/5$ and $1/7$. Comparing these energies to the corresponding composite fermion energies we find the same result — the composite fermion energy gap is consistently smaller than the corresponding Laughlin energy gap. Comparing these results to the extrapolated exact diagonalization results of Fano et al. [9] we find that the composite fermion energy gaps are consistently closer to the 'exact' results. However, as for $\nu = 1/3$, the differences are slight, and the main conclusion is that both wave function provide an adequate description of the physical quasielectron.

This paper is organized as follows. In Sec. II we review the formulation of the two dimensional electron gas on the Haldane sphere and introduce a procedure for projecting wave functions on the sphere into the lowest Landau level. In Sec. III we review both the composite fermion construction of the quasielectron state, which requires the projection developed in Sec. II, as well as the Laughlin quasielectron wave function. Finally, Sec. IV contains a summary of our results.

II. THE TWO DIMENSIONAL ELECTRON GAS ON THE HALDANE SPHERE

A. One Particle and Landau Level Projection

We begin by reviewing the problem of one spin polarized electron confined to move on the surface of a sphere of radius R with a magnetic monopole at its center. Following convention we let

$$2S = \frac{4\pi R^2 B}{hc/e} = 2\left(\frac{R}{l_0}\right)^2$$

denote the number of flux quanta piercing the surface of the sphere due to the monopole. The Hamiltonian describing this particle is then

$$T = \frac{|\mathbf{\Lambda}|^2}{2mR^2} \tag{1}$$

where $\mathbf{\Lambda} = \mathbf{r} \times (-i\hbar\nabla + e\mathbf{A}(\mathbf{r}))$ and $\nabla \times \mathbf{A} = B\hat{\mathbf{r}}$ on the surface of the sphere. In what follows we work in the Wu-Yang gauge [10] for which

$$\mathbf{A} = \frac{\hbar S}{eR}\frac{1-\cos\theta}{\sin\theta}\mathbf{e}_\phi \tag{2}$$

and use the complex stereographic coordinate $z = x + iy = \cot\frac{\theta}{2}e^{i\phi}$. The eigenfunctions of T in this gauge are the monopole harmonics [10]

$$Y_{Slm} = M_{Slm}\left(\frac{1}{1+|z|^2}\right)^S z^{S+m} P_{l-s}^{S+m,S-m}\left(\frac{1-|z|^2}{1+|z|^2}\right) \tag{3}$$

where

$$M_{Slm} = \left(\frac{2l+1}{4\pi}\frac{(l-s)!(l+s)!}{(l-m)!(l+m)!}\right)^{1/2}, \tag{4}$$

$P_n^{\alpha,\beta}$ is a Jacobi polynomial, $l = S, S{+}1, S{+}2, \cdots$, and for a given l, $m = -l, -l{+}1, \cdots, l{-}1, l$. If we let $n = S - l$ then the energies of these states are

$$E_n = \hbar\omega\left(n + \frac{1}{2} + \frac{n(n+1)}{2S}\right), \qquad n = 0, 1, 2, \cdots \tag{5}$$

and n is the spherical Landau level index.

It will be necessary in what follows to project general wave functions onto the lowest Landau level ($n = 0$) Hilbert space. Following Girvin and Jach [11] we now introduce a general procedure for performing such a projection on the sphere. First note that the lowest Landau level wave functions in the Wu-Yang gauge are

$$\psi_m = \left(\frac{2S+1}{4\pi}\begin{pmatrix} 2S \\ S+M \end{pmatrix}\right)^{1/2} \left(\frac{1}{1+|z|^2}\right)^S z^p, \qquad p = 0, 1, \cdots, 2S+1.$$

The Hilbert space of lowest Landau level wave functions on the sphere then corresponds to wave functions of the form

$$\psi(z, z^*) = \left(\frac{1}{1+|z|^2}\right)^S f(z) \tag{6}$$

where $f(z)$ is a polynomial of degree up to $2S+1$ in z.

The differential area element on the surface of the sphere in terms of the stereographic coordinates x and y is

$$dA = \frac{4R^2}{(1+|z|^2)^2} dx dy \tag{7}$$

and so the scalar product between any two polynomials f and g in this Hilbert space is

$$\langle f, g\rangle = \int \frac{4R^2}{(1+|z|^2)^{2S+2}} f^* g \; dx \; dy. \tag{8}$$

With this definition of the scalar product it is straightforward to derive the following identity by repeatedly integrating by parts, exploiting the fact that $\frac{d}{dz}(f(z)^*) = 0$,

$$\left\langle f, \frac{d^n}{dz^n} g\right\rangle = \frac{(2S+2)!}{(2S+2-n)!}\left\langle f, \left(\frac{z^*}{1+|z|^2}\right)^n g\right\rangle. \tag{9}$$

This result immediately implies the following spherical generalization of the $z^* \to d/dz$ rule of Girvin and Jach to the sphere,

$$\left(\frac{z^*}{1+|z|^2}\right)^n \Rightarrow \frac{(2S+2-n)!}{(2S+2)!}\frac{d^n}{dz^n} \tag{10}$$

Thus, to project any polynomial $f(z, z^*/(1+|z|^2))$ into the lowest Landau level one simply orders each term so that the $z^*/(1+|z|^2)$'s all sit to the left, then replaces these factors by derivatives with respect to z according to (10).

B. N particles and the fractional QHE

The Hamiltonian for N spin polarized electrons on the sphere interacting via the Coulomb repulsion is then

$$H = \sum_{i=1}^{N} T_i + V. \tag{11}$$

The interaction energy V is

$$V = \sum_{i<j} \frac{e^2}{r_{ij}} + \frac{1}{2}\frac{Q^2}{R} - \frac{NeQ}{R}, \tag{12}$$

where r_{ij} is the chord distance between a given pair of electrons on the sphere. Here the sphere is assumed to have a uniform compensating charge density with total charge Q. When considering a homogeneous state the appropriate background charge is $Q = Ne$ for which

$$V = \sum_{i<j} \frac{e^2}{r_{ij}} - \frac{1}{2}\frac{N^2e^2}{R}. \tag{13}$$

The spherical analog of the Laughlin state [3] at $\nu = 1/q$ occurs when $q(N-1) = 2S$ and in the Wu-Yang gauge is

$$\psi_{gs} = \prod_k \left(\frac{1}{1+|z_k|^2}\right)^S \prod_{i<j}(z_i - z_j)^q. \tag{14}$$

The quasihole wave function corresponding to a single charge $-e/q$ defect at the top of the sphere ($z = 0$) is

$$\psi_{qh} = \prod_k \left(\frac{1}{1+|z_k|^2}\right)^S z_k \prod_{i<j}(z_i - z_j)^q. \tag{15}$$

III. QUASIELECTRON WAVE FUNCTIONS

A. Composite Fermion Quasielectron Wave Function

According to Jain's composite fermion approach [4] a given fractional QHE wave function at filling fraction $\nu = p/(kp+1)$ where $k = 2, 4, \cdots$ is found by first constructing the corresponding *integer* QHE wave function at $\nu_{CF} = p$ and then multiplying by a Jastrow factor which ties k vortices to each electron. The state is then explicitly projected into the lowest Landau level. Denoting the Slater determinant corresponding to the effective integer QHE state, with the overall $\prod(1+|z|^2)^{-\tilde{S}}$ factor removed, as Φ_{CF}, the corresponding fractional QHE states are

$$\psi = P_{\mathrm{LLL}} \prod \left(\frac{1}{1+|z|^2} \right)^S \prod_{i<j} (z_i - z_j)^n \Phi_{CF} \tag{16}$$

where P_{LLL} is the projection operator onto the lowest Landau level.

For the $\nu_{CF} = 1$ ground state the Vandermonde determinant

$$\Phi_{CF} = \begin{vmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^{N-1} \\ \vdots & & & & \vdots \\ 1 & z_N & z_N^2 & \cdots & z_N^{N-1} \end{vmatrix} = \prod_{i<j} (z_i - z_j) \tag{17}$$

corresponding to one filled 'pseudo'-Landau level of composite fermions, gives for ψ the Laughlin wave function (14) for $\nu = 1/q$ where $q = k+1$. If we remove a composite fermion from the lowest 'pseudo'-Landau level then

$$\Phi_{CF} = \begin{vmatrix} z_1 & z_1^2 & \cdots & z_1^{N-1} \\ \vdots & & & \vdots \\ z_N & z_N^2 & \cdots & z_N^{N-1} \end{vmatrix} = \prod_i z_i \prod_{i<j} (z_i - z_j) \tag{18}$$

and ψ is identical to (15) and describes a state in which a single quasihole sits at the top of the sphere.

We now consider the quasielectron wave function constructed using this approach. If we introduce a composite fermion into the first excited 'pseudo'-Landau level then

$$\phi_{CF} = \begin{vmatrix} 1 & z_1 & \dots & z_1^{N-2} & \frac{z_1^*}{1+|z_1|^2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & z_N & \dots & z_N^{N-2} & \frac{z_N^*}{1+|z_N|^2} \end{vmatrix}. \tag{19}$$

and the corresponding physical electron wave function is

$$\psi_{qe}^{CF} = P_{\mathrm{LLL}} \prod_k \left(\frac{1}{1+|z_k|^2} \right)^S \prod_{i<j} (z_i - z_j)^{q-1} \begin{vmatrix} 1 & z_1 & \dots & z_1^{N-2} & \frac{z_1^*}{1+|z_1|^2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & z_N & \dots & z_N^{N-2} & \frac{z_N^*}{1+|z_N|^2} \end{vmatrix}. \tag{20}$$

This wave function can be projected into the lowest Landau level as follows. First rewrite ψ_{qh}^{CF} by pulling the Jastrow factor and the projection operator into the last column of the determinant,

$$\psi_{qh}^{CF} = \prod_k \left(\frac{1}{1+|z_k|^2} \right)^S \begin{vmatrix} 1 & z_1 & \dots & z_1^{N-2} & P_{\mathrm{LLL}} \frac{\overline{z}_1}{1+|z_1|^2} \prod_{i<j} (z_i - z_j)^{q-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & z_N & \dots & z_N^{N-2} & P_{\mathrm{LLL}} \frac{\overline{z}_N}{1+|z_N|^2} \prod_{i<j} (z_i - z_j)^{q-1} \end{vmatrix}. \tag{21}$$

This can be done here because the cofactor associated with the n^{th} element of the N^{th} column does not contain z_n. Thus, when we do the projection we need only project each element of the matrix separately. Following the procedure outlined in Sec. IIA this projection gives

$$P_{\mathrm{LLL}} \frac{\overline{z}_1}{1+|z_1|^2} \prod_{i<j} (z_i - z_j)^{q-1} = \frac{1}{2S+2} \frac{\partial}{\partial z_1} \prod_{i<j} (z_i - z_j)^{q-1} = \frac{q-1}{2S+2} \prod_{i<j} (z_i - z_j)^{q-1} \sum_{i \neq 1} \frac{1}{z_1 - z_i}. \tag{22}$$

Thus ψ_{qe}^{CF} can be rewritten, up to an irrelevant normalization constant, to give

$$\psi_{qe}^{CF} = \prod_k \left(\frac{1}{1+|z_k|^2} \right)^S \prod_{i<j} (z_i - z_j)^{q-1} \begin{vmatrix} 1 & z_1 & \dots & z_1^{N-2} & \sum_{i \neq 1} \frac{1}{z_1 - z_i} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & z_N & \dots & z_N^{N-2} & \sum_{i \neq N} \frac{1}{z_N - z_i} \end{vmatrix}. \tag{23}$$

One can simplify things further by expanding the determinant in a cofactor expansion down the N^{th} column. The cofactors are then all Vandermonde determinants and the final expression for the Jain quasielectron is

$$\psi_{qe}^{CF} = \sum_n \prod_{k \neq n} \frac{1}{z_k - z_n} \sum_{l \neq n} \frac{1}{z_l - z_n} \psi_{\mathrm{gs}}. \tag{24}$$

In this form $|\psi|^2$ can be sampled by usual variational Monte Carlo techniques with each Monte Carlo step taking order N instructions, rather than N^2 for a usual determinant.

B. Laughlin Quasielectron Wave Function

The generalization to the spherical geometry of the quasielectron wave function introduced by Laughlin is

$$\psi_{qp} = \left(\prod_k \left(\frac{1}{1+|z_k|^2} \right)^S \frac{\partial}{\partial z_k} \right) \prod_{i<j} (z_i - z_j)^q \tag{25}$$

Straightforward Monte Carlo sampling of $|\psi_{qp}|^2$ is not possible because of the explicit derivatives with respect to the electron coordinates. To compute the energy of this state we therefore follow the procedure of Morf and Halperin [5,6] which, for completeness, we review below. A more detailed discussion can be found in [6].

Following [2] we first take the absolute square of the wave function to obtain

$$\begin{aligned} |\psi_{qp}|^2 &= \left(\prod_k \left(\frac{1}{1+|z_k|^2} \right)^{2S} \frac{\partial}{\partial z_k} \frac{\partial}{\partial z_k^*} \right) \prod_{i<j} |z_i - z_j|^{2q} \\ &= \left(\prod_k \left(\frac{1}{1+|z_k|^2} \right)^{2S} \frac{1}{4} \nabla_k^2 \right) \prod_{i<j} |z_i - z_j|^{2q}. \end{aligned} \tag{26}$$

The expectation value of any operator O depending only on the coordinates $\{(x_i, y_i)\}$ is then

$$\langle O \rangle = \frac{\int \left(\frac{1}{1+|z_k|^2} \right)^{2S+2} O \nabla_k^2 \prod_{i<j} |z_i - z_j|^{2q} \prod_i dx_i dy_i}{\int \left(\frac{1}{1+|z_k|^2} \right)^{2S+2} \nabla_k^2 \prod_{i<j} |z_i - z_j|^{2q} \prod_i dx_i dy_i} \tag{27}$$

which, after integrating by parts twice in the numerator and the denominator can be rewritten

$$\langle O \rangle = \frac{\int P(z_1, ..., z_N) \tilde{O} \prod_i dx_i dy_i}{\int P(z_1, ..., z_N) \prod_i dx_i dy_i} \tag{28}$$

where

$$P = \left(\prod_k \left(\frac{1}{1+|z_i|^2} \right)^{2S+4} \left(|z_k|^2 - \frac{1}{2S+2} \right) \right) \prod_{i<j} |z_i - z_j|^{2q} \tag{29}$$

and

$$\tilde{O} = \frac{\prod_j \nabla_j^2 \left(\frac{1}{1+|z_j|^2}\right)^{2S+4} O}{\prod_j \nabla_j^2 \left(\frac{1}{1+|z_j|^2}\right)^{2S+4}} \tag{30}$$

The chord distance between any two points on the sphere is given by $r_{ij} = R|z_i - z_j|\sqrt{(1+|z_i|^2)(1+|z_j|^2)}$ and so for the Coulomb interaction the operator O is

$$V_{\text{Coul.}} = \frac{e^2}{R} \frac{\sqrt{1+|z_i|^2}\sqrt{1+|z_j|^2}}{|z_i - z_j|} \tag{31}$$

It is then straightforward to compute $\tilde{O}$ and evaluate (28) by usual variational Monte Carlo techniques.

IV. RESULTS

The excitation energies of isolated quasielectron and quasihole states have been obtained using the trial wave functions reviewed in Secs. II and III. Following [6] we have computed the *proper* energies, meaning that the relevant ground state energies are computed with monopole strength $2S = q(N-1)$ and background charge $Q = Ne$, while the energy of the quasielectron (quasihole) excitations are obtained keeping R and N fixed and decreasing (increasing) the monopole strength according to $S \to S - 1/2$ ($S \to S + 1/2$). In addition, following [3] and [9], we have shifted the background charge when computing the quasielectron (quasihole) energies taking $Q \to Q - e/m$ ($Q \to Q + e/m$), in order to compensate the charge density of the *bulk* of the wave function. This eliminates a finite size correction of $\pm - (e/m)^2 1/R \sim O(\frac{1}{\sqrt{N}})$, a correction which was not included in [6] and which may account for the slightly different results obtained here. Our results for the proper energies of the quasiholes, the Laughlin quasielectrons, and the composite fermion quasielectrons for filling fractions $\nu = 1/3, 1/5$ and $1/7$ are shown plotted vs. $1/N$ in Fig. 1.

The quasihole energy gaps obtained here, $\Delta^{q.h.}$, extrapolated to the $N \to \infty$ limit are given in Table I, together with the $\nu = 1/3$ result of Morf and Halperin [6] and the extrapolated exact diagonalization results of Fano et al. [9]. As stated above, the discrepancy between our results and those of Morf and Halperin is most likely due to the $O(1/sqrtN)$ finite size correction we have eliminated by modifying the background charge density.

The quasielectron energy gaps obtained using both the Laughlin trial state, $\Delta_L^{q.e.}$, and the composite fermion state, $\Delta_{CF}^{q.e.}$ are givin in Table II. Again, in comparing the present $\nu = 1/3$ result for $\Delta_L^{q.e.}$ with those of [6] we note a slight discrepancy which we attribute to the $1/\sqrt{N}$ finite size correction we have included here. For $\nu = 1/3, 1/5$, and $1/7$ the composite fermion excitation energy $\Delta_{q.e.}^{CF}$ is consistently about %10 lower than the Laughlin excitation energy, as can be seen clearly in Fig. 1. Note that the composite fermion result is also in slightly better agreement with the extrapolated exact diagonalization results of Fano et al. [9].

Finally, Table III gives results for the total energy gap for creating a well separated quasielectron quasihole pair, $\Delta = \Delta^{q.h.} + \Delta^{q.e.}$. The results are again compared with [6] for the Laughlin energy gap as well as those of [7] for the composite fermion energy gap. For $\nu = 1/3$ our extrapolated energy gap using the Laughlin quasielectron is $\Delta_L = 0.110(2)$, roughly %20 larger than the earlier estimate of Morf and Halperin [6]. Our improved calculation gives a Laughlin energy gap which is slightly larger than the corresponding energy gap computed using the composite fermion quasielectron, consistent with the results of Girlich and Hellmund [8] using the short-range V_1 model. We find here that for the Coulomb interaction the energy gaps obtained using these two wave functions are essentially the same and so we conclude that that both wave functions provide adequate descriptions of the physical quasielectron, though the energy gap obtained using the composite fermion quasielectron is closer to extrapolated exact-diagonalization results.

To summarize, the quasielectron and quasihole excitation energies in the fractional QHE have been calculated for $\nu = 1/3, 1/5$ and $1/7$ by variational Monte Carlo. Results have been obtained using the quasielectron states originally proposed by Laughlin, as well as the fully projected composite fermion quasielectron states proposed by Jain. We have improved on earlier estimates [5,6] of the excitation energies of the Laughlin states at $\nu = 1/3$ in order to show that the composite-fermion energy gap is actually slightly lower than the Laughlin energy gap, consistent with the results of Girlich and Hellmund [8]. Results for the energy gap using Laughlin's quasielectron for $\nu = 1/5$ and $1/7$, obtained here for the first time, show that as for $\nu = 1/3$ for Coulomb interactions the energy gaps obtained using the Laughlin and composite fermion quasielectron wave functions are essentially the same, though the energy gaps obtained using the composite fermion quasielectrons is slightly smaller, and closer to extrapolated exact-diagonalization results of Fano et al. [9], than those obtained using the Laughlin quasielectron.

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TABLES

TABLE I. Quasihole energy for the fractional QHE with $\nu = 1/3$, $1/5$, and $1/7$. The Monte Carlo results of Morf and Halperin [6] and extrapolated exact diagonalization results of Fano et al. [9] are given for comparison.

ν	$\Delta^{\text{q.h.}}$	$\Delta^{\text{q.h.}}(Ref.[1])$	$\Delta^{\text{q.h.}}(Ref.[2])$
1/3	0.0279(12)	0.0224(16)	0.0264
1/5	0.0092(6)	—	0.0071
1/7	0.0038(4)	—	—

TABLE II. Quasielectron energy for the fractional QHE with $\nu = 1/3$, $1/5$, and $1/7$. The Monte Carlo results of Morf and Halperin [6] and extrapolated exact diagonalization results of Fano *et al.* [9] are given for comparison.

ν	$\Delta^{\text{q.e.}}_{\text{CF}}$	$\Delta^{\text{q.e.}}_{\text{L}}$	$\Delta^{\text{q.e.}}_{\text{L}}(Ref.\ [6])$	$\Delta^{\text{q.e.}}(Ref.\ [9])$
1/3	0.0779(10)	0.0825(12)	0.075(5)	0.0772
1/5	0.0166(6)	0.0191(6)	—	0.0173
1/7	0.0063(4)	0.0070(5)	—	—

TABLE III. Total gap for the fractional QHE with $\nu = 1/3$, $1/5$, and $1/7$. The Monte Carlo results of Morf and Halperin [6] and Bonesteel [7], and extrapolated exact diagonalization results of Fano *et al.* [9] are given for comparison.

ν	Δ_{CF}	Δ_{L}	$\Delta_{\text{L}}(Ref.\ [6])$	$\Delta_{\text{CF}}(\ [7])$	$\Delta(Ref.\ [9])$
1/3	0.1058(16)	0.1104(17)	0.092(4)	0.106(3)	0.1036(2)
1/5	0.0258(9)	0.0283(9)	—	0.025(3)	0.0244(3)
1/7	0.0101(6)	0.0108(6)	—	0.011(3)	—

FIGURES

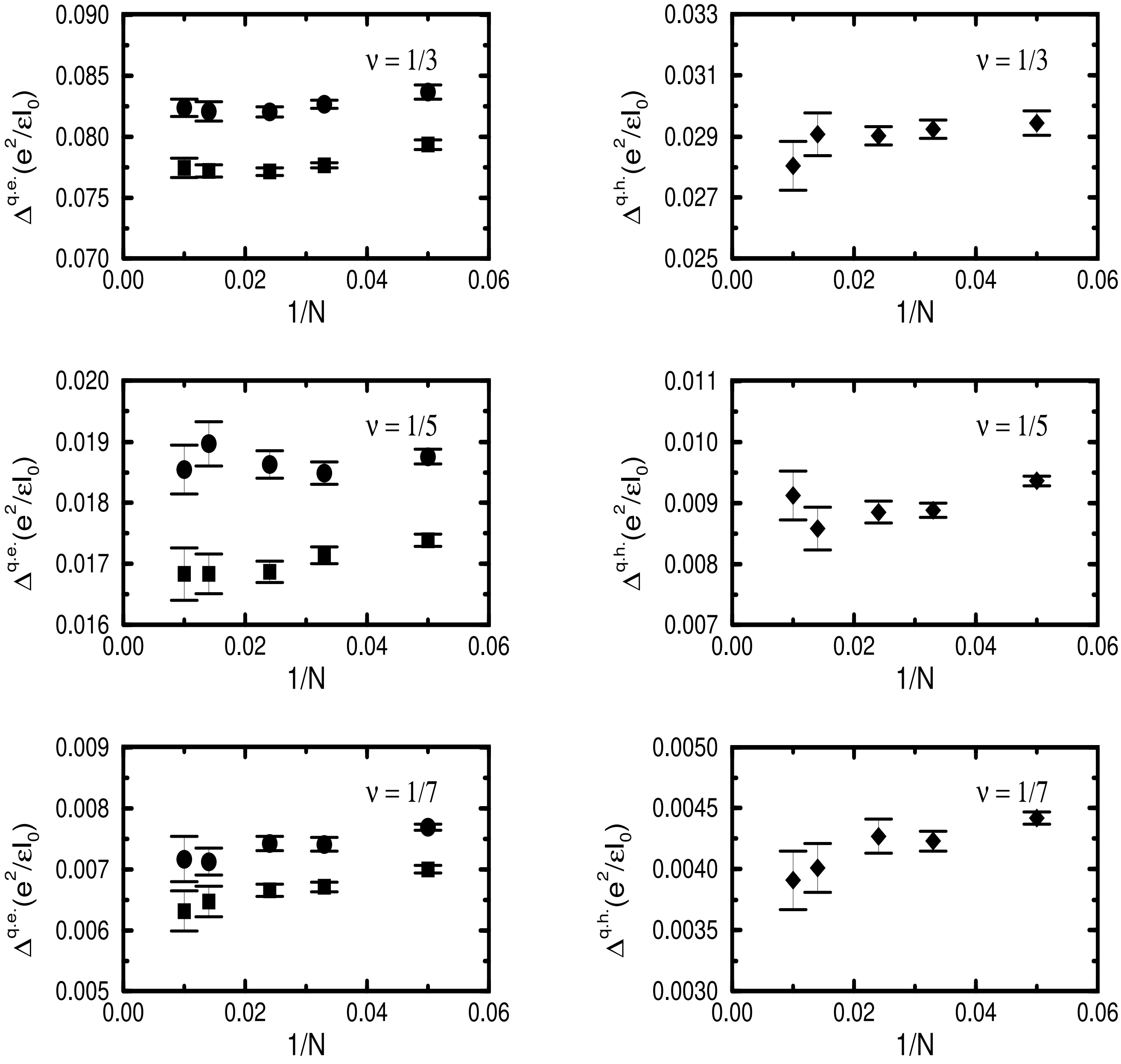

FIG. 1. Proper energies for the Laughlin quasielectron (solid circles), composite fermion quasielectron (solid squares), and quasihole (solid diamonds) for filling fractions $\nu = 1/3$, $1/5$ and $1/7$, plotted vs. $1/N$.