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QUANTUM COMPUTING WITH NON-ABELIAN QUASIPARTICLES

N. E. BONESTEEL,* L. HORMOZI and G. ZIKOS

Department of Physics and NHMFL, Florida State University, 1800 E. Paul Dirac Dr., Tallahassee, FL 32310, USA
** bonestee@magnet.fsu.edu*

S. H. SIMON

Bell Laboratories, Lucent Technologies
Murray Hill, New Jersey 07974, USA

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In topological quantum computation quantum information is stored in exotic states of matter which are intrinsically protected from decoherence, and quantum operations are carried out by dragging particle-like excitations (quasiparticles) around one another in two space dimensions. The resulting quasiparticle trajectories define world-lines in three dimensional space-time, and the corresponding quantum operations depend only on the topology of the braids formed by these world-lines. We describe recent work showing how to find braids which can be used to perform arbitrary quantum computations using a specific kind of quasiparticle (those described by the so-called Fibonacci anyon model) which are thought to exist in the experimentally observed $\nu = 12/5$ fractional quantum Hall state.

Keywords: Quantum computation; Quantum Hall effect; non-Abelian statistics.

1. Introduction

One of the most remarkable phenomena associated with the fractional quantum Hall effect is that of *fractionalization.* This refers to the fact that when an electron is added to a fractional quantum Hall state it can break apart into fractionally charged quasiparticle excitations with exotic quantum statistics.

Over the past several years a growing body of theoretical work has begun to suggest an entirely unexpected potential application of this effect. It has been shown that a particularly exotic class of fractionalized quasiparticles — quasiparticles which obey so-called non-Abelian statistics[1] — can, at least in principle, be used to build a universal quantum computer.[2,3]

In a quantum computer it will be necessary for quantum bits (qubits) — two-level quantum systems spanned by states denoted $|0\rangle$ and $|1\rangle$ — to first be initialized, then manipulated coherently to carry out a particular quantum algorithm (typically

*Corresponding author.

using a set of quantum gates which act on one or two qubits at a time), and finally measured. The biggest obstacle to building such a device is the inevitability of errors and loss of quantum coherence of the qubits due to their coupling to the environment during the computation.

It is on this point that non-Abelian fractional quantum Hall states may be of some help. These states have the property that when quasiparticles are present, with their positions fixed and kept sufficiently far apart, there is a degenerate Hilbert space of states whose dimensionality grows exponentially with the number of quasiparticles.[1] Furthermore, these degenerate states are characterized by "topological" quantum numbers. This means that no local measurement can be used to determine what state the system is in — only global measurements can do this. These states are therefore essentially invisible to the environment and so have a built-in protection against decoherence, making them an ideal place to store qubits.

To apply unitary operations to this degenerate Hilbert space (i.e. to "quantum compute") quasiparticle excitations can be dragged around one another in two space dimensions so that their world-lines form "braids" in 2+1 dimensional space-time. The quasiparticles are presumed to satisfy non-Abelian statistics, meaning that whenever two are exchanged a matrix operation which depends only on the topology of the space-time path used to carry out the exchange is applied to the Hilbert space. After carrying out a complicated braid involving many quasiparticles, the resulting unitary operation will likewise depend only on the topology of this braid. The resulting intrinsic "fault tolerance" is another appealing feature of this form of quantum computation, which is known as topological quantum computation.[2,3]

There is a strong theoretical case[4,5] that the $\nu = 5/2$ fractional quantum Hall state is described by the non-Abelian state originally proposed by Moore and Read.[1] More recently, a compelling case has also been made[6] that the $\nu = 12/5$ state (recently detected experimentally[7]) is also a non-Abelian state, this time described by the so-called $k = 3$ "parafermion" state proposed by Read and Rezayi.[8] While it has been shown that universal quantum computation is not possible using quasiparticles in the $\nu = 5/2$ Moore-Read state, it *is* possible using quasiparticles in the $\nu = 12/5$ Read-Rezayi state.[3,9] The non-Abelian properties of the relevant quasiparticle excitations of this state are described by the so-called Fibonacci anyon model[10,11] and we will refer to these quasiparticles as Fibonacci anyons.

In this paper we review an explicit procedure we have recently developed for constructing braiding patterns for Fibonacci anyons which can be used to carry out any desired quantum algorithm. For more detailed discussions of this and related work see Refs. 12 and 13.

2. Encoded Qubits and Elementary Braid Matrices

Fibonacci anyons carry a topological quantum number which we refer to as q-spin (in the language of quantum groups[9,14] q-spin is a "q-deformed" spin quantum number).

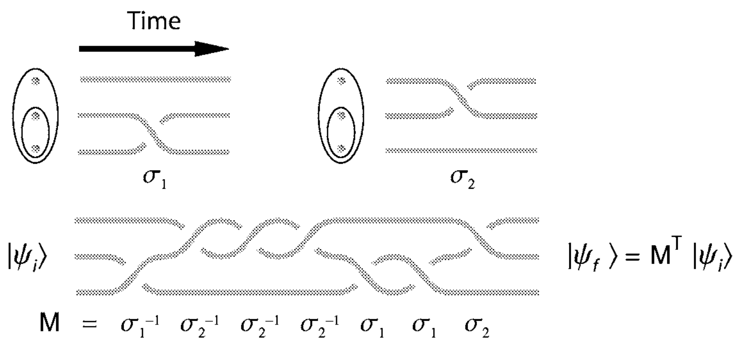

Fig. 1. Elementary braid operations acting on three quasiparticles and the evaluation of a general braid in terms of these elementary operations. The solid dots represent Fibonacci anyons and the ovals enclosing these dots play the same role as the parenthesis in the notation $((\bullet,\bullet),\bullet)$ used in the text (in this case they represent a basis choice). σ_1 and σ_2 are the elementary braid matrices given in the text.

Each Fibonacci anyon has a q-spin of 1, and when more than one Fibonacci anyon is present there are only two possible values for their total q-spin, 0 or 1. The rules for combining q-spin (the so-called fusion rules) are simply $0\otimes 0 = 0$, $1\otimes 0 = 0\otimes 1 = 1$, and $1\otimes 1 = 0\oplus 1$. These rules can be viewed as q-deformed versions of the usual triangle rule for combining ordinary spin.

The Fibonacci fusion rules fix the structure of the Hilbert space of many Fibonacci anyons. For example, the fusion rule $1\otimes 1 = 0\oplus 1$ implies the Hilbert space of two Fibonacci anyons is two dimensional, spanned by states we can represent as $(\bullet,\bullet)_0$ and $(\bullet,\bullet)_1$. Here we have introduced a notation in which the symbol $\bullet$ represents a Fibonacci anyon, and any group of Fibonacci anyons enclosed in parenthesis are in a q-spin eigenstate with the eigenvalue given by the subscript of the parenthesis. Likewise, the Hilbert space of three Fibonacci anyons is three dimensional and spanned by the states $|0\rangle = ((\bullet,\bullet)_0,\bullet)_1$, $|1\rangle = ((\bullet,\bullet)_1,\bullet)_1$ and $|NC\rangle = ((\bullet,\bullet)_1,\bullet)_0$ (the state $((\bullet,\bullet)_0,\bullet)_0$ is forbidden by the fusion rule $0\otimes 1 = 1$). Following Freedman et al.[3] we will encode qubits using the total q-spin 1 sector of this three quasiparticle Hilbert space, taking the two states labeled $|0\rangle$ and $|1\rangle$ as the logical 0 and 1 states of the qubit. The total q-spin 0 state $|NC\rangle$ is then what we will refer to as a "noncomputational" state. We note in passing that for N Fibonacci anyons the Hilbert space degeneracy is the $(N+1)^{st}$ Fibonacci number.[8]

Fig. 1 shows the elementary braid operations σ_1 and σ_2 that can be applied to three quasiparticles. Associated with each of these braid operations there is a matrix which acts on the three-dimensional Hilbert space of the three Fibonacci anyons. These matrices are[9,10]

$$\sigma_1 = \left(\begin{array}{cc|c} e^{-i4\pi/5} & 0 & \\ 0 & e^{i3\pi/5} & \\ \hline & & e^{i3\pi/5} \end{array}\right); \quad \sigma_2 = \left(\begin{array}{cc|c} -\tau e^{-i\pi/5} & \sqrt{\tau}e^{-i3\pi/5} & \\ \sqrt{\tau}e^{-i3\pi/5} & -\tau & \\ \hline & & e^{i3\pi/5} \end{array}\right), \quad (1)$$

where the upper left 2×2 blocks of these matrices act on the computational qubit

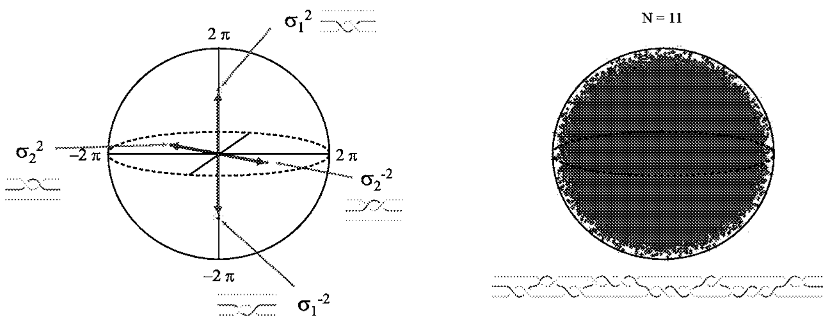

Fig. 2. Rotation vectors corresponding to single-qubit rotations carried out by three quasiparticle weaves. Each single-qubit rotation U is parameterized by its rotation vector $\vec{\alpha}$ where $U = \exp(i\vec{\alpha} \cdot \vec{\sigma}/2)$ (here $\vec{\sigma}$ is the Pauli matrix vector, not to be confused with the braid matrices σ_1 and σ_2). (Left) The $\vec{\alpha}$ vectors corresponding to the four elementary weaving operators σ_1^2, σ_2^2, σ_1^{-2}, and σ_2^{-2}. These vectors can be represented as points inside a solid sphere of radius 2π. (Right) The set of all points corresponding to $\vec{\alpha}$ vectors for braids built out of up to eleven elementary weaving operations (or twenty-two elementary interchanges). A representative braid of this length is also shown.

space ($|0\rangle$ and $|1\rangle$) while the lower right matrix element is a phase factor which is applied to the state $|NC\rangle$. The form of these matrices are essentially fixed (up to a choice of chirality and overall Abelian phase factors which are irrelevant for quantum computation) by certain consistency conditions dictated by the fusion rules.[10,15] To compute the unitary operation produced by an arbitrary braid involving three strands one then simply expresses the braid as a sequence of elementary braid operations and multiplies the corresponding matrices (σ_1, σ_2 and their inverses) to obtain the net transformation, as shown in Fig. 1.

From the form of σ_1 and σ_2 we see that such braiding will never lead to transitions from the computational qubit space (spanned by $|0\rangle$ and $|1\rangle$) into the noncomputational state $|NC\rangle$. This reflects a general feature of non-Abelian quasiparticles — no amount of braiding within a group of quasiparticles will change its total q-spin.

As we envision carrying out a single-qubit operation on one of these encoded qubits it is convenient to consider a restricted class of braids known as *weaves* — braids in which only one quasiparticle moves. It is known that any operation which can be carried out by a braid can also be carried out by a weave.[16] Figure 2 shows the rotation vectors $\vec{\alpha}$ corresponding to single-qubit rotations $U_{\vec{\alpha}} = \exp(i\vec{\alpha} \cdot \vec{\sigma}/2)$ generated by four elementary weave operations — σ_1^2, σ_2^2, σ_1^{-2} and σ_2^{-2}. These squared braid matrices describing weave operations in which the middle quasiparticle is woven once around either the top or bottom quasiparticle in either a clockwise or counterclockwise sense.

Because the number of topologically distinct braids grows exponentially with braid length, and because the operators $\sigma_1^{\pm 2}$ and $\sigma_2^{\pm 2}$ generate a group which is dense in $SU(2)$,[11] the set of distinct operations which can be carried out by braids rapidly fills the space of all single-qubit rotations, as is also shown in Fig. 2. By carrying out brute force searches over braids with up to 46 elementary braid operations we typically find braids which approximate a desired target gate to a distance (measured by operator norm[17]) of $\sim 10^{-3}$.

As the demand for braid accuracy increases the search space of braids continues to grow exponentially and brute force search rapidly becomes infeasible. Fortunately the Solovay-Kitaev algorithm[17,18] can be used to efficiently improve the accuracy of any given braid with the length of the braid growing only polylogarithmically with the inverse of the distance to the desired target gate. Thus, a combination of brute force search and the Solovay-Kitaev algorithm can be used to efficiently find braids which carry out any desired single-qubit operation to any required accuracy.[13]

3. Two-Qubit Gates

To carry out arbitrary quantum computations it is of course not enough to be able to carry out single-qubit operations — it is also necessary to be able to carry out at least one two-qubit entangling gate.[17] The standard choice for such a gate is a controlled-NOT (CNOT) gate, a gate in which the state of one qubit (the target) is flipped if and only if another qubit (the control) is in the state $|1\rangle$.

For qubits encoded using triplets of Fibonacci anyons two-qubit gates involve braiding six quasiparticles and at least two new problems arise when we try to find braids which carry out such gates. First, because quasiparticles will no longer just be braided within qubits there will in general be transitions out of the computational space into the noncomputational state $|NC\rangle$. Second, the space of unitary operations which act on the six quasiparticle Hilbert space is $SU(5) \oplus SU(8)$ — a Lie group which has eighty seven generators as opposed to three for the $SU(2)$ case. This means the gate search space will no longer be the simple three-dimensional sphere shown in Fig. 2, but a much larger eighty-seven dimensional space, making direct brute force search problematic.

We therefore consider a restricted class of braids — those in which a pair of quasiparticles from one qubit (the control) are woven around the quasiparticles forming the second qubit (the target) (see Fig. 3). This construction exploits the fact that if this pair of quasiparticles has total q-spin 0 (which, if we choose the pair properly, will correspond to the control qubit being in the state $|0\rangle$) then the resulting operation is trivial (i.e. the identity), but if it is 1 (corresponding to the control qubit being in the state $|1\rangle$) the result is to carry out some nontrivial unitary operation on the system. We therefore automatically get a controlled operation of some kind.

The problem is still difficult because we must find a weaving pattern for which the controlled operation does not induce transitions out of the computational space (i.e. does not induce transitions to the state $|NC\rangle$). Furthermore, this construction involves weaving a pair of quasiparticles through three other quasiparticles, i.e. it involves braiding *four* objects. Ideally we would like to reduce this to three, since,

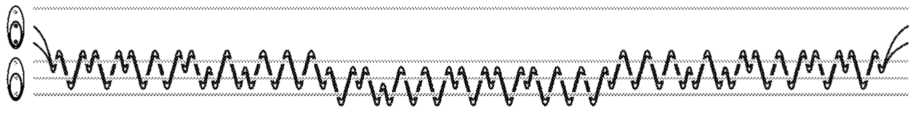

Fig. 3. Braiding pattern which approximates a CNOT gate (up to single qubit operations) for two qubits encoded using triplets of Fibonacci anyons. A pair of quasiparticles from the top (control) qubit is woven through the quasiparticles forming the bottom (target) qubit in such a way that the state of the bottom qubit is acted on by the operator $i\sigma_x$ if and only if the top qubit is in the state $|1\rangle$ (see text).

as shown above, we can efficiently find braids involving three objects. We therefore adopt a "divide and conquer" approach, reducing the problem of finding a two-qubit gate to that of finding a sequence of three stranded braids, turning the problem of constructing a CNOT gate to a series of smaller, tractable problems. While this procedure does not yield the optimal braid of a given length which approximates a desired two-qubit gate, we believe it does lead to the most accurate two-qubit gates which can be obtained for a fixed amount of classical computing power.

Figure 3 shows a two-qubit braid which approximates a CNOT gate (up to single qubit operations), constructed using the "divide and conquer" method just described. In this braid the pair is first woven through the top two quasiparticles of the target qubit in such a way that the unitary operation it carries out when the pair has total q-spin 1 is approximately the identity. We refer to this initial weave as an injection weave because it has the effect of injecting the target qubit with the pair in such a way that the new target has the same q-spin quantum numbers as the original target. We then weave the pair within the injected target (the middle part of the braid) in order to carry out a NOT operation (or more precisely an $i\sigma_x$ operation), and finally extract the pair using the inverse of the injection weave. If the control qubit is in the state $|0\rangle$ the pair has total q-spin 0 and this braid does nothing, as explained above, but if the control qubit is in the state $|1\rangle$ the pair has q-spin 1 and a NOT operation is carried out on the target. The resulting gate is therefore a CNOT gate (up to single qubit rotations which are needed to eliminate the phase factor of i in the $i\sigma_x$ operation applied to the target). As for single-qubit gates, this two-qubit gate can be improved to whatever accuracy is required using the Solovay-Kitaev algorithm with only a polylogarithmic increase in length.

Since any quantum algorithm can be decomposed into a series of single-qubit operations and CNOT gates, our explicit construction of braiding patterns which carry out these gates provides a direct proof that topological quantum computation is possible for Fibonacci anyons. This proof complements the more abstract mathematical proof of Freedman et al.,[3] and has the added benefit of providing a practical recipe for translating any quantum algorithm into a braid.

4. Conclusion

Needless to say, the technological requirements for physically carrying out these braids will be quite demanding. But, given the inherent advantages of this form of quantum computation — namely the built-in protection against decoherence due to the topological nature of the Hilbert space, and the intrinsic fault tolerance of braiding — it is not out of the question that, in the long run, quantum computers may actually one day be built out of fractional quantum Hall matter. Even if this does not happen, the realization that the quantum order associated with the fractional quantum Hall effect can be sufficiently rich to carry out universal quantum computation has provided a surprising link between the physics of the fractional quantum Hall effect and the theory of quantum computation.

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