NAME:

ELECTRICITY AND MAGNETISM II

Second Midterm Exam (Spring 2017)

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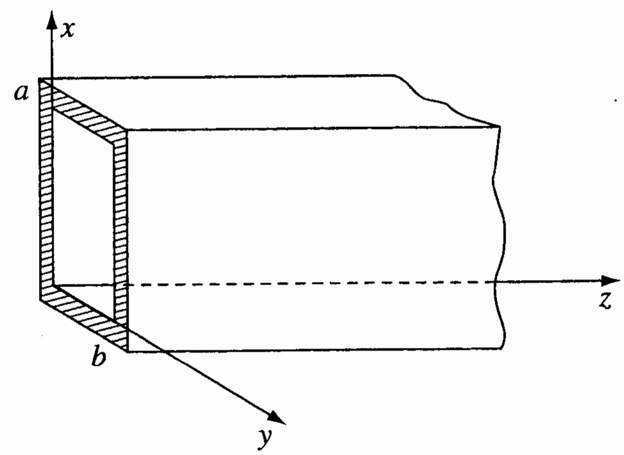
1 : (13 points)

Consider the propagation of TE waves in a rectangular wave guide with height a and width b $(a > b)$. The problem reduces to solving

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right] B_z(x,y) = 0 \ ,$$

subject to the boundary condition $B^{\perp} = 0$.



(a) Solve the equation by separation of variables. (3 pts.)

(b) Reduce the number of independent constants of the solution by applying the boundary conditions and obtain $B_z(x,y)$. Recall that

$$B_x = \frac{ik}{(\omega/c)^2 - k^2}\frac{\partial B_z}{\partial x} \quad \text{and} \quad B_y = \frac{ik}{(\omega/c)^2 - k^2}\frac{\partial B_z}{\partial y} \quad . \text{(3 pts.)}$$

(c) What is the cutoff frequency for the mode with m (n) nodes along the x (y) direction. (1 pt.)

(d) Find the smallest cutoff frequency. (1 pt.)

(e) State the frequency range for propagation in the z direction. (1 pt.)

(f) State the wave number of the propagation in the z direction. (1 pt.)

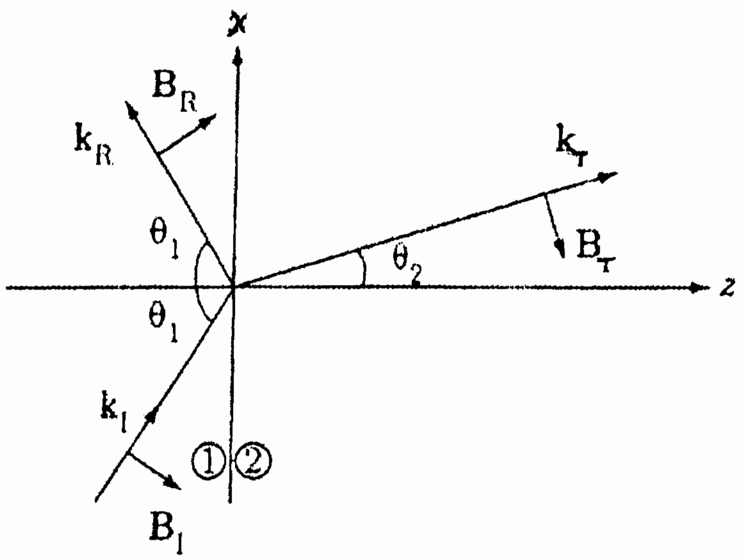
(g) Obtain the wave velocity (phase velocity), v, of the wave. (1 pt.)

(h) Obtain the group velocity, v_g. The energy propagates with v_g. (1 pt.)

(i) Verify that the group velocity is less than c. Is the phase velocity smaller or larger than c? (1 pt.)

 # 2 : (12 points)

Consider a plane wave with the polarization perpendicular to the plane of incidence. Here z is normal to the surface and xz is the plane of incidence, i.e. the electric field is in the y direction (see figure).



(i) $\epsilon_1 \left(\tilde{\mathbf{E}}_{0_I} + \tilde{\mathbf{E}}_{0_R}\right)_z = \epsilon_2 \left(\tilde{\mathbf{E}}_{0_T}\right)_z$

(ii) $\left(\tilde{\mathbf{B}}_{0_I} + \tilde{\mathbf{B}}_{0_R}\right)_z = \left(\tilde{\mathbf{B}}_{0_T}\right)_z$

(iii) $\left(\tilde{\mathbf{E}}_{0_I} + \tilde{\mathbf{E}}_{0_R}\right)_{x,y} = \left(\tilde{\mathbf{E}}_{0_T}\right)_{x,y}$

(iv) $\dfrac{1}{\mu_1} \left(\tilde{\mathbf{B}}_{0_I} + \tilde{\mathbf{B}}_{0_R}\right)_{x,y} = \dfrac{1}{\mu_2} \left(\tilde{\mathbf{B}}_{0_T}\right)_{x,y}$

(a) Write down the expressions for the incoming, reflected and transmitted electric and magnetic fields. (3 pts.)

(b) Impose the boundary conditions given above, and obtain the Fresnel equations for $\tilde{E}_{0R}$ and $\tilde{E}_{0T}$. You may use the law of refraction: $\sin(\theta_2)/\sin(\theta_1) = v_2/v_1$. Express the Fresnel equations in terms of $\alpha = \cos(\theta_2)/\cos(\theta_1)$ and $\beta = \mu_1 v_1/\mu_2 v_2$. (6 pts.)

(c) Compute the reflection, $R = I_R/I_I$, and transmission, $T = I_T/I_I$, coefficients, and check that they add up to 1. The intensity I_I for the incoming wave is defined as $I_I = (1/2)\epsilon_1 v_1 E_{0I}^2 \cos(\theta_I)$ and similar definitions hold for I_R and I_T. (3 pts.)

Solution of problem #1

(a) $$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega}{c}\right)^2-k^2\right]B_z(x,y)=0$$

$B_z(x,y)=X(x)\,Y(y)$: $\quad Y\frac{d^2X}{dx^2}+X\frac{d^2Y}{dy^2}+\left[\left(\frac{\omega}{c}\right)^2-k^2\right]XY=0$

divide by XY, then

$$\frac{1}{X}\frac{d^2X}{dx^2}=-k_x^2\ ,\quad \frac{1}{Y}\frac{d^2Y}{dy^2}=-k_y^2\ ,\quad -k_x^2-k_y^2+\left(\frac{\omega}{c}\right)^2-k^2=0$$

solution : $\quad X(x)=A\sin(k_x x)+B\cos(k_x x)$

$\qquad\qquad\quad Y(y)=C\sin(k_y y)+D\cos(k_y y)$

(b) Boundary conditions: $B_x=0$ for $x=0,a$ and $B_y=0$ for $y=0,b$.

Hence, $\frac{\partial B_z}{\partial x}=\frac{\partial B_z}{\partial y}=0$ at boundary or $\frac{dX}{dx}=0=\frac{dY}{dy}$

Consequently $A=C=0$ and $k_x=\frac{m\pi}{a}\ ,\ m=0,1,2,\ldots$

$\qquad\qquad\qquad\qquad\qquad\quad\ k_y=\frac{n\pi}{b}\ ,\ n=0,1,2,\ldots$

$$B_z=B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

(c) $$k^2=\left(\frac{\omega}{c}\right)^2-\pi\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]=\frac{1}{c^2}\left[\omega^2-\omega_{mn}^2\right]$$

$$\omega_{mn}=\pi c\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}$$

(d) since $a>b$, the smallest cutoff frequency is $\omega_{10}=\frac{c\pi}{a}$

(e) frequency range for propagation: $\omega>\omega_{mn}$

(f) wavenumber: $k=\frac{1}{c}\sqrt{\omega^2-\omega_{mn}^2}$

(g) phase velocity : $$v=\frac{\omega}{k}=\frac{\omega c}{\sqrt{\omega^2-\omega_{mn}^2}}=\frac{c}{\sqrt{1-\left(\frac{\omega_{mn}}{\omega}\right)^2}}$$

(h) group velocity : $$v_g=\frac{d\omega}{dk}=\frac{1}{\frac{dk}{d\omega}}=\frac{c}{\omega/\sqrt{\omega^2-\omega_{mn}^2}}=c\sqrt{1-\left(\frac{\omega_{mn}}{\omega}\right)^2}$$

(i) $v_g = c\sqrt{1-\left(\frac{\omega}{\omega_{mn}}\right)^2} < c$ ✓

$v = \frac{c}{\sqrt{1-\left(\frac{\omega}{\omega_{mn}}\right)^2}} > c$ ✓

Solution of problem # 2

(a) incoming fields
$$\begin{cases} \tilde{\vec{E}}_I = \tilde{E}_{0I}\, e^{i(\vec{k}_I\cdot\vec{r}-\omega t)}\,\hat{y} \\ \tilde{\vec{B}}_I = \tilde{E}_{0I}\frac{1}{v_1}\, e^{i(\vec{k}_I\cdot\vec{r}-\omega t)}\left(-\cos\theta_1\,\hat{x} + \sin\theta_1\,\hat{z}\right) \end{cases}$$

reflected fields
$$\begin{cases} \tilde{\vec{E}}_R = \tilde{E}_{0R}\, e^{i(\vec{k}_R\cdot\vec{r}-\omega t)}\,\hat{y} \\ \tilde{\vec{B}}_R = \frac{1}{v_1}\tilde{E}_{0R}\, e^{i(\vec{k}_R\cdot\vec{r}-\omega t)}\left(\cos\theta_1\,\hat{x} + \sin\theta_1\,\hat{z}\right) \end{cases}$$

transmitted fields
$$\begin{cases} \tilde{\vec{E}}_T = \tilde{E}_{0T}\, e^{i(\vec{k}_T\cdot\vec{r}-\omega t)}\,\hat{y} \\ \tilde{\vec{B}}_T = \tilde{E}_{0T}\frac{1}{v_2}\, e^{i(\vec{k}_T\cdot\vec{r}-\omega t)}\left(-\cos\theta_2\,\hat{x} + \sin\theta_2\,\hat{z}\right) \end{cases}$$

(b) Boundary conditions: For $z=0$, $\vec{k}_I\cdot\vec{r}-\omega t = \vec{k}_R\cdot\vec{r}-\omega t = \vec{k}_T\cdot\vec{r}-\omega t$;
hence the exponentials can be dropped when applying boundary conditions

(i) $0=0$ (trivial because $\vec{E}\perp\hat{z}$)

(iii) $\tilde{\vec{E}}_{0I} + \tilde{\vec{E}}_{0R} = \tilde{\vec{E}}_{0T}$ (since $\vec{E}\parallel\hat{y}$), $\boxed{\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}}$

(ii) $\frac{1}{v_1}\tilde{E}_{0I}\sin\theta_1 + \frac{1}{v_1}\tilde{E}_{0R}\sin\theta_1 = \frac{1}{v_2}\tilde{E}_{0T}\sin\theta_2$ or

$\tilde{E}_{0I} + \tilde{E}_{0R} = \frac{v_1\sin\theta_2}{v_2\sin\theta_1}\tilde{E}_{0T} = \tilde{E}_{0T}$ (same as (iii))

(iv) $\frac{1}{\mu_1}\left[\frac{1}{v_1}\tilde{E}_{0I}(-\cos\theta_1) + \frac{1}{v_1}\tilde{E}_{0R}\cos\theta_1\right] = \frac{1}{\mu_2 v_2}\tilde{E}_{0T}(-\cos\theta_2)$

$\boxed{\tilde{E}_{0I} - \tilde{E}_{0R} = \frac{\mu_1 v_1}{\mu_2 v_2}\frac{\cos\theta_2}{\cos\theta_1}\tilde{E}_{0T} = \alpha\beta\,\tilde{E}_{0T}}$

Solving for $\tilde{E}_{0R}$ and $\tilde{E}_{0T}$:

$$\tilde{E}_{0T} = \frac{2}{1+\alpha\beta}\tilde{E}_{0I}$$

$$\tilde{E}_{0R} = \frac{1-\alpha\beta}{1+\alpha\beta}\tilde{E}_{0I}$$

The reflected wave is in-phase if $\alpha\beta < 1$ and 180° out of phase if $\alpha\beta > 1$.

<u>Fresnel equations</u> for polarization perpendicular to the plane of incidence

$$\boxed{E_{0T} = \frac{2}{1+\alpha\beta} E_{0I} \quad , \quad E_{0R} = \left|\frac{1-\alpha\beta}{1+\alpha\beta}\right| E_{0I}}$$

(c) Reflection and transmission coefficients

$$R = \frac{I_R}{I_I} = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2$$

$$T = \frac{I_T}{I_I} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\alpha\left(\frac{E_{0T}}{E_{0I}}\right)^2 = \overbrace{\frac{\varepsilon_2 \mu_2}{\varepsilon_1 \mu_1}}^{=\left(\frac{v_1}{v_2}\right)^2}\frac{\mu_1 v_2}{\mu_2 v_1}\alpha\left(\frac{2}{1+\alpha\beta}\right)^2 =$$

$$= \underbrace{\frac{\mu_1 v_1}{\mu_2 v_2}}_{=\beta}\alpha\left(\frac{2}{1+\alpha\beta}\right)^2 = \alpha\beta\left(\frac{2}{1+\alpha\beta}\right)^2$$

$$R + T = 1 \quad \checkmark$$