Relativistic Mechanics

proper time $d\tau$; ordinary time dt

proper velocity $\vec{\eta} \equiv \frac{d\vec{l}}{d\tau}$; ordinary velocity $\vec{u} = \frac{d\vec{l}}{dt}$

$$\vec{\eta} = \frac{1}{\sqrt{1-u^2/c^2}}\,\vec{u}$$

$\vec{\eta}$ is spatial part of four-vector $\eta^\mu \equiv \frac{dx^\mu}{d\tau}$, $\eta^0 = \frac{dx^0}{d\tau} = c\frac{dt}{d\tau}$

η^μ invariant under Lorentz transformation $= \frac{c}{\sqrt{1-u^2/c^2}}$

$$\bar{\eta}^\mu = \Lambda^\mu_{\ \nu}\,\eta^\nu \qquad (4\text{-velocity})$$

relativistic momentum $\vec{p} \equiv m\vec{\eta} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}}$

$p^\mu \equiv m\eta^\mu$ (4-vector) ; $p^0 = m\eta^0 = \frac{mc}{\sqrt{1-u^2/c^2}}$

Einstein: $E = \frac{mc^2}{\sqrt{1-u^2/c^2}}$; p^μ : energy-momentum 4-vector

Rest energy : $E_{rest} = mc^2$

$$E_{kin} \equiv E - E_{rest} = mc^2\left(\frac{1}{\sqrt{1-u^2/c^2}} - 1\right) \simeq \frac{1}{2}mu^2 + \frac{3}{8}\frac{mu^4}{c^2} + \dots$$

Invariant quantity : same value in all inertial systems

Conserved quantity : same value before and after some process

In every closed system, the total relativistic energy and momentum are conserved

Mass : invariant but not conserved

Energy : conserved but not invariant

velocity : neither conserved nor invariant

electric charge : conserved and invariant

$$p^{\mu}p_{\mu} = -(p^0)^2 + (\vec{p}\cdot\vec{p}) = \frac{-m^2c^2 + m^2u^2}{1-u^2/c^2} = -m^2c^2 = -\left(\frac{E}{c}\right)^2 + \vec{p}^{\,2}$$

$$\boxed{E^2 - p^2c^2 = m^2c^4}$$

Examples:

① kinetic energy $\rightarrow$ rest energy

② rest energy $\rightarrow$ kinetic energy

③ photons

④ pion decay: $E_{before} = m_\pi c^2$ $\quad \vec{P}_{before} = 0$

$E_{after} = E_\mu + E_\nu$ $\quad \vec{P}_{after} = \vec{p}_\mu + \vec{p}_\nu$

$\vec{p}_\nu = -\vec{p}_\mu$, $E_\mu + E_\nu = m_\pi c^2$, $E_\nu = |\vec{p}_\nu| c$, $|\vec{p}_\mu| = \frac{1}{c}\sqrt{E_\mu^2 - m_\mu^2 c^4}$

$$E_\mu + \sqrt{E_\mu^2 - m_\mu^2 c^4} = m_\pi c^2 \longrightarrow E_\mu = \frac{(m_\pi^2 + m_\mu^2)c^2}{2m_\pi}$$

Collisions

⑤ Compton scattering: momentum cons.

$$\begin{cases} p_e \sin\phi = p_p \sin\theta, \quad p_p = \dfrac{E}{c} \\ \dfrac{E_0}{c} = p_p\cos\theta + p_e\cos\phi = \dfrac{E}{c}\cos\theta + p_e\sqrt{1-\left(\dfrac{E}{p_e c}\sin\theta\right)^2} \end{cases}$$

$$p_e^2c^2 = (E_0 - E\cos\theta)^2 + E^2\sin^2\theta = E_0^2 - 2EE_0\cos\theta + E^2$$

energy cons. $E_0 + mc^2 = E + E_e = E + \sqrt{m^2c^4 + p_e^2c^2} =$

$$= E + \sqrt{m^2c^4 + E_0^2 - 2E_0E\cos\theta + E^2}$$

solving for E $\quad E = \dfrac{1}{(1-\cos\theta)/mc^2 + 1/E_0}$; $\quad E = h\nu = \dfrac{hc}{\lambda}$

$$\boxed{\lambda = \lambda_0 + \frac{h}{mc}(1-\cos\theta)}$$

$\frac{h}{mc}$ Compton wavelength for electron magnetic dipole: rectangle with current I, electric field $\vec{E}$

$(I = \lambda u)$ $\quad I = \frac{QN_+}{l} u_+ = \frac{QN_-}{l} u_-$, $N_+ u_+ = \frac{Il}{Q}$

classical momentum: $\quad p_{cl} = MN_+ u_+ - MN_- u_- = M\frac{Il}{Q} - M\frac{Il}{Q} = 0$

no momentum since the loop is not moving

relativistic momentum: (now $\vec{p} = \gamma M \vec{u}$)

$$p_{rel} = \gamma_+ MN_+ u_+ - \gamma_- MN_- u_- = \frac{MIl}{Q}(\gamma_+ - \gamma_-) \neq 0$$

gain in KE = work by electric field

$Mc^2(\gamma_+ - \gamma_-) = QEw$, $\quad p_{rel} = \frac{IlEw}{c^2}$

induced magnetic dipole moment: $m = Ilw$ points into the page, $\vec{p}$ to the right, then

$$\boxed{\vec{p} = \frac{1}{c^2}(\vec{m} \times \vec{E})}$$

thus, a magnetic dipole at rest in an electric field carries mechanical linear momentum, although the loop is at rest

- hidden momentum is strictly relativistic
- precisely cancels the electromagnetic momentum stored in the fields

problem # 20.5 $\quad \vec{p}_{el\text{-}m} = -\frac{1}{c^2}(\vec{m} \times \vec{E})$

Magnetism as a Relativistic Phenomenon

Example: Fig. 12.34a : no electric force

Fig. 12.34b : $v_\pm = \dfrac{v \mp u}{1 \mp \frac{vu}{c^2}}$, $v_- > v_+$

Lorentz contraction: $\lambda_\pm = \pm(\gamma_\pm)\lambda_0$, $\gamma_\pm = \dfrac{1}{\sqrt{1 - v_\pm^2/c^2}}$

λ_0 = charge density in its own rest frame

λ = charge density in S

$$\lambda = \gamma \lambda_0 \quad , \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$\gamma_\pm = \frac{1}{\sqrt{1 - \frac{1}{c^2}\frac{(v \mp u)^2}{(1 \mp vu/c^2)^2}}} = \frac{c^2 \mp uv}{\sqrt{(c^2 \mp uv)^2 - c^2(v \mp u)^2}} = \frac{c^2 \mp uv}{\sqrt{(c^2 - v^2)(c^2 - u^2)}}$$

$$= \frac{1 \mp uv/c^2}{\sqrt{(1 - \frac{v^2}{c^2})(1 - \frac{u^2}{c^2})}} = \gamma \frac{1 \mp uv/c^2}{\sqrt{1 - u^2/c^2}}$$

Net line charge in $\bar{S}$: $\lambda_{tot} = \lambda_+ + \lambda_- = \lambda_0(\gamma_+ - \gamma_-) = -\dfrac{2\lambda u v}{c^2\sqrt{1 - u^2/c^2}}$

due to unequal Lorentz contraction for positive and negative line charges

$$E = \frac{\lambda_{tot}}{2\pi\varepsilon_0 s} \quad , \quad \bar{F} = qE = -\frac{\lambda v}{\pi\varepsilon_0 c^2 s}\,\frac{qu}{\sqrt{1 - u^2/c^2}}$$

We now transform this force back to the S system where q is at rest:

$$F = \sqrt{1 - u^2/c^2}\,\bar{F} = -\frac{\lambda v}{\pi\varepsilon_0 c^2}\cdot\frac{qu}{s} = -qu\left(\frac{\mu_0 I}{2\pi s}\right) \qquad (I = 2\lambda v)$$

$$= -quB$$

<u>Force</u>: purely electrical in $\bar{S}$, but purely magnetic in S

Transform $\vec{E}$-fields from S to $\bar{S}$

capacitor at rest in S_0 : $\vec{E}_0 = \frac{\sigma_0}{\varepsilon_0}\hat{y}$

capacitor moving with v_0 in S : $\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{y}$ (Fig. 12.35)

l is Lorentz-contracted, Q and w not changed ; $\gamma_0 = \frac{1}{\sqrt{1-v_0^2/c^2}}$

$\sigma = \gamma_0 \sigma_0$; $\vec{E}^{\perp} = \gamma_0 \vec{E}_0^{\perp}$ (perpendicular to motion)

Fig. 12.36 : d is Lorentz contracted, but not Q, l, w.

$E^{\parallel} = E_0^{\parallel}$ (parallel to motion)

Example : $\vec{E}$-field of a point charge in uniform motion.

S_0 : q at rest, $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_0^2}\hat{r}_0$ or

$$E_{x_0} = \frac{1}{4\pi\varepsilon_0}\frac{q x_0}{(x_0^2+y_0^2+z_0^2)^{3/2}},\quad E_{y_0} = \frac{1}{4\pi\varepsilon_0}\frac{q y_0}{(x_0^2+y_0^2+z_0^2)^{3/2}},\quad E_{z_0} = \frac{1}{4\pi\varepsilon_0}\frac{q z_0}{(x_0^2+y_0^2+z_0^2)^{3/2}}$$

S : moves along x at speed v_0 relative to S_0

$$E_x = E_{x_0} = \frac{1}{4\pi\varepsilon_0}\frac{q x_0}{(x_0^2+y_0^2+z_0^2)^{3/2}},\quad E_y = \gamma_0 E_{y_0} = \frac{1}{4\pi\varepsilon_0}\frac{\gamma_0 q y_0}{(x_0^2+y_0^2+z_0^2)^{3/2}},$$

$$E_z = \gamma_0 E_{z_0} = \frac{1}{4\pi\varepsilon_0}\frac{\gamma_0 q z_0}{(x_0^2+y_0^2+z_0^2)^{3/2}},\quad \gamma_0 = \frac{1}{\sqrt{1-v_0^2/c^2}}$$

Write (x_0, y_0, z_0) in terms of coordinates in S; inverse Lorentz transformation

$x_0 = \gamma_0(x + v_0 t) = \gamma_0 R_x$, $y_0 = y = R_y$, $z_0 = z = R_z$

here $\vec{R}$ is vector from q to P (in S_0).

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{\gamma_0 q \vec{R}}{(\gamma_0^2 R^2\cos^2\theta + R^2\sin^2\theta)^{3/2}} = \frac{1}{4\pi\varepsilon_0}\frac{q/\gamma_0^2}{\left[1+\left(\frac{1}{\gamma_0^2}-1\right)\sin^2\theta\right]^{3/2}}\frac{\hat{R}}{R^2}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{q\,(1 - v_0^2/c^2)}{\left[1 - \frac{v_0^2}{c^2}\sin^2\theta\right]^{3/2}} \frac{\hat{R}}{R^2}$$

– Same result we had obtained from retarded potentials!!
The balancing of factors γ_0 in the numerator make $\vec{E} \parallel \vec{R}$!

Transform $\vec{E}$ and $\vec{B}$ fields from S to $\bar{S}$

charge at rest in S_0, S moves with v_0 (along x) and
$\bar{S}$ moves with v relative to S (along x).

Back to parallel plate capacitor: surface current $\vec{K}_{\pm} = \mp \sigma v_0 \hat{x}$
(Fig. 12.35b)

$$E_y = \frac{\sigma}{\varepsilon_0}, \quad B_z = -\mu_0 \sigma v_0$$

(from Ampère's law; rectangular loop perpendicular to surface and the current, same result for both plates)

Now introduce system $\bar{S}$; the fields in $\bar{S}$ are ($\bar{v}$ relative to S_0)

$$\bar{E}_y = \frac{\bar{\sigma}}{\varepsilon_0}, \quad \bar{B}_z = -\mu_0 \bar{\sigma} \bar{v}, \quad \bar{v} = \frac{v + v_0}{1 + v v_0/c^2}, \quad \bar{\gamma} = \frac{1}{\sqrt{1 - \bar{v}^2/c^2}}$$

$$\bar{\sigma} = \bar{\gamma}\sigma_0$$

Express $\bar{E}$ and $\bar{B}$ in terms of E and B (in S):

$$\bar{E}_y = \frac{\bar{\gamma}}{\gamma_0} \frac{\sigma}{\varepsilon_0}, \quad \bar{B}_z = -\frac{\bar{\gamma}}{\gamma_0} \mu_0 \sigma \bar{v}$$

We show that

$$\frac{\bar{\gamma}}{\gamma_0} = \frac{\sqrt{1 - v_0^2/c^2}}{\sqrt{1 - \bar{v}^2/c^2}} = \frac{\sqrt{1 - v_0^2/c^2}}{\sqrt{1 - \frac{1}{c^2}\left(\frac{v + v_0}{1 + v v_0/c^2}\right)^2}} = \frac{(1 + v v_0/c^2)\sqrt{1 - v_0^2/c^2}}{\sqrt{\left(1 + \frac{v v_0}{c^2}\right)^2 - \left(\frac{v + v_0}{c}\right)^2}} =$$

$$= \frac{(1 + v v_0/c^2)\sqrt{1 - v_0^2/c^2}}{\sqrt{(1 - v^2/c^2)(1 - v_0^2/c^2)}} = \gamma\left(1 + \frac{v v_0}{c^2}\right), \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Now

$$\boxed{\bar{E}_y} = \gamma\left(1+\frac{v v_0}{c^2}\right)\frac{\sigma}{\varepsilon_0} = \gamma\left(\frac{\sigma}{\varepsilon_0} + \frac{v}{c^2\varepsilon_0\mu_0}\mu_0\sigma v_0\right) = \boxed{\gamma\left(E_y - vB_z\right)}$$

$$\boxed{\bar{B}_z} = -\gamma\left(1+\frac{v v_0}{c^2}\right)\mu_0\sigma\bar{v} = -\gamma\left(1+\frac{v v_0}{c^2}\right)\mu_0\sigma\,\frac{v+v_0}{1+\frac{v v_0}{c^2}} = \gamma\left(-\mu_0\sigma v_0 - \mu_0\sigma v\right)$$

$$= \gamma\left(B_z - \mu_0\varepsilon_0 v\frac{\sigma}{\varepsilon_0}\right) = \boxed{\gamma\left(B_z - \frac{v}{c^2}E_y\right)}$$

Similarly, to get the transformation of E_z and B_y we repeat the procedure with the same capacitor parallel to the xy plane (instead of xz plane).

in S: $E_z = \frac{\sigma}{\varepsilon_0}$, $B_y = \mu_0\sigma v_0$

the remainder is identical to previous case with $E_y \to E_z$ and $B_z \to -B_y$

$$\boxed{\bar{E}_z = \gamma\left(E_z + vB_y\right)} \quad , \quad \boxed{\bar{B}_z = \gamma\left(B_y + \frac{v}{c^2}E_z\right)}$$

For E_x we have already shown that $E_0^{\parallel} = E^{\parallel}$ so that

$$\boxed{\bar{E}_x = E_x}$$

solenoid (Fig. 12.40): $\boxed{\bar{B}_x = B_x}$

Transformation of E_y and B_z:

$$\bar{E}_y = \gamma(E_y - v B_z) \qquad \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

Transformation of E_z and B_y:

$$\bar{E}_z = \gamma(E_z + v B_y) \qquad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right)$$

Transformation for E_x and B_x:

$$\bar{E}_x = E_x \quad , \quad \bar{B}_x = B_x$$

Special cases:

(1) if $\vec{B} = 0$ in S $\quad \vec{\bar{B}} = \gamma \frac{v}{c^2}(E_z \hat{y} - E_y \hat{z}) = \frac{v}{c^2}(\bar{E}_z \hat{y} - \bar{E}_y \hat{z}) =$

$$= -\frac{1}{c^2}(\vec{v} \times \vec{\bar{E}}) \qquad (\vec{v} = v\hat{x})$$

(2) if $\vec{E} = 0$ in S $\quad \vec{\bar{E}} = -\gamma v(B_z \hat{y} - B_y \hat{z}) = -v(\bar{B}_z \hat{y} - \bar{B}_y \hat{z}) =$

$$= \vec{v} \times \vec{\bar{B}}$$

Example: magnetic field of point charge q moving at constant $\vec{v}$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q(1 - v^2/c^2)}{\left[1 - \frac{v^2}{c^2}\sin^2\theta\right]^{3/2}} \frac{\hat{R}}{R^2} \quad , \quad \vec{B} = \frac{1}{c^2}(\vec{v} \times \vec{E})$$

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{q v (1 - v^2/c^2)}{\left[1 - \frac{v^2}{c^2}\sin^2\theta\right]^{3/2}} \sin\theta \frac{\hat{\phi}}{R^2}$$

nonrelativistic limit $(v^2 \ll c^2)$

$$\vec{E} \simeq \frac{1}{4\pi\varepsilon_0} \frac{q}{R^2}\hat{R} \quad \text{(Coulomb)} \quad , \quad \vec{B} \simeq \frac{\mu_0}{4\pi} q \frac{\vec{v} \times \hat{R}}{R^2} \quad \text{(Biot-Savart)}$$

Field tensor

antisymmetric, second-rank tensor, $t^{\mu\nu} = -t^{\nu\mu}$

transformation: $\bar{t}^{\mu\nu} = \Lambda^{\mu}_{\lambda} t^{\lambda\sigma} \Lambda^{\nu}_{\sigma}$

6 independent entries (diagonal is zero): 3 for $\vec{E}$ and 3 for $\vec{B}_z$

field tensor:

$$F^{\mu\nu} = \begin{Bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{Bmatrix}$$

dual tensor: $\vec{E}/c \to \vec{B}$, $\vec{B} \to -\vec{E}/c$

$$G^{\mu\nu} = \begin{Bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{Bmatrix}$$

transformation of sources: $\rho = Q/V$, $\vec{J} = \rho \vec{u}$

in terms of proper charge density $\rho_0 = Q/V_0$, V_0 is rest volume

$V = V_0 \sqrt{1 - u^2/c^2}$, $\underbrace{\rho = \rho_0 \frac{1}{\sqrt{1-u^2/c^2}}, \quad \vec{J} = \rho_0 \frac{\vec{u}}{\sqrt{1-u^2/c^2}}}_{\text{current density 4-vector}}$ $J^{\mu} = \rho_0 \eta^{\mu}$

$J^{\mu} = (c\rho, J_x, J_y, J_z)$

continuity equation $\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$ (charge conservation)

$\vec{\nabla} \cdot \vec{J} = \sum_{i}^{3} \frac{\partial J^i}{\partial x^i}$, $\frac{\partial \rho}{\partial t} = \frac{1}{c} \frac{\partial J^0}{\partial t} = \frac{\partial J^0}{\partial x^0}$; $\boxed{\frac{\partial J^{\mu}}{\partial x^{\mu}} = 0}$

Maxwell's equations:

$$\boxed{\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu \quad , \quad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0}$$

if $\mu = 0$
$$\frac{\partial F^{0\nu}}{\partial x^\nu} = \frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3} =$$
$$= \frac{1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) = \frac{1}{c}\left(\vec{\nabla}\cdot\vec{E}\right) = \mu_0 J^0 = \mu_0 c \rho$$
$$\boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}}$$
Gauss's law

if $\mu = 1$
$$\frac{\partial F^{1\nu}}{\partial x^\nu} = \frac{\partial F^{10}}{\partial x^0} + \frac{\partial F^{11}}{\partial x_1} + \frac{\partial F^{12}}{\partial x_2} + \frac{\partial F^{13}}{\partial x_3} = -\frac{1}{c^2}\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} =$$
$$= \left(-\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\times\vec{B}\right)_x = \mu_0 J^1 = \mu_0 J_x$$
$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}}$$
Ampère's law + Maxwell correction

if $\mu = 0$
$$\frac{\partial G^{0\nu}}{\partial x^\nu} = \frac{\partial G^{00}}{\partial x^0} + \frac{\partial G^{01}}{\partial x^1} + \frac{\partial G^{02}}{\partial x^2} + \frac{\partial G^{03}}{\partial x^3} =$$
$$= \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \boxed{\vec{\nabla}\cdot\vec{B} = 0}$$
3rd Maxwell equation

if $\mu = 1$
$$\frac{\partial G^{1\nu}}{\partial x^\nu} = \frac{\partial G^{10}}{\partial x^0} + \frac{\partial G^{11}}{\partial x^1} + \frac{\partial G^{12}}{\partial x^2} + \frac{\partial G^{13}}{\partial x^3} = -\frac{1}{c}\frac{\partial B_x}{\partial t} - \frac{1}{c}\frac{\partial E_z}{\partial y} + \frac{1}{c}\frac{\partial E}{\partial z}$$
$$= -\frac{1}{c}\left(\frac{\partial \vec{B}}{\partial t} + \vec{\nabla}\times\vec{E}\right)_x = 0$$
$$\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$
Faraday's law

Minkowski force:

$$\boxed{K^{\mu} = q\,\eta_{\nu} F^{\mu\nu}}$$

if $\mu = 1$

$$K^1 = q\,\eta_{\nu} F^{1\nu} = q\left(-\eta_0 F^{10} + \eta_1 F^{11} + \eta_2 F^{12} + \eta_3 F^{13}\right) =$$

$$= q\left[\frac{-c}{\sqrt{1-u^2/c^2}}\left(-\frac{E_x}{c}\right) + \frac{u_y}{\sqrt{1-u^2/c^2}}\left(B_z\right) + \frac{u_z}{\sqrt{1-u^2/c^2}}\left(-B_y\right)\right]$$

$$= \frac{q}{\sqrt{1-u^2/c^2}}\left[\vec{E} + (\vec{u}\times\vec{B})\right]_x$$

$$\vec{K} = \frac{q}{\sqrt{1-u^2/c^2}}\left[\vec{E} + (\vec{u}\times\vec{B})\right], \qquad \vec{F} = \sqrt{1-\frac{u^2}{c^2}}\,\vec{K}$$

(ordinary vs. proper force)

$$\boxed{\vec{F} = q\left[\vec{E} + (\vec{u}\times\vec{B})\right]}$$

Lorentz force

if $\mu = 0$

$$K^0 = \frac{q}{c}\gamma\,\vec{u}\cdot\vec{E} = \frac{1}{c}\frac{dW}{d\tau}$$

W = energy of particle

$$d\tau = \frac{1}{\gamma}dt \Rightarrow \boxed{\frac{dW}{dt} = q(\vec{u}\cdot\vec{E})}$$

power delivered to particle = force $q\vec{E}$ × velocity $\vec{u}$