Solution of problem # 12.1

(a) $V = \frac{Q}{C} = IR$; positive I means the charge on the capacitor decreases ; hence $\frac{dQ}{dt} = -I = -\frac{Q}{RC}$. Hence,

$Q(t) = Q_0 e^{-t/RC}$, but $Q_0 = Q(0) = CV_0$, so

$\boxed{Q(t) = CV_0 e^{-t/RC}}$. It follows that

$$\boxed{I(t) = -\frac{dQ}{dt} = CV_0 \frac{1}{RC} e^{-t/RC} = \frac{V_0}{R} e^{-t/RC}}$$

(b) Initial energy stored in the capacitor: $\boxed{W = \frac{1}{2} C V_0^2}$

The energy delivered to the resistor is

$$\int_0^\infty dt\, P = \int_0^\infty dt\, I^2 R = \frac{V_0^2}{R} \int_0^\infty dt\, e^{-2t/RC} = \frac{V_0^2}{R} \left[-\frac{RC}{2} e^{-2t/RC} \right]_0^\infty =$$

$$= \frac{V_0^2 C}{2} \quad \checkmark$$ equal to the initial energy stored in the capacitor

(c) $V_0 = IR + \frac{Q}{C}$, now positive I means Q is increasing.

$$\frac{dQ}{dt} = \frac{V_0}{R} - \frac{Q}{RC} = \frac{1}{RC}(V_0 C - Q) \quad ; \quad \frac{dQ}{Q - V_0 C} = -\frac{1}{RC} dt$$

integrate

$$\ln(Q - CV_0) = -\frac{1}{RC} t + \text{const} \quad ; \quad Q(t) = CV_0 + k e^{-t/RC} ,$$

but $Q(0) = 0 \Rightarrow k = -CV_0$ and $\boxed{Q(t) = CV_0 \left(1 - e^{-t/RC}\right)}$

$$I(t) = \frac{dQ}{dt} = CV_0 \frac{1}{RC} e^{-t/RC} = \frac{V_0}{R} e^{-t/RC} ,$$ as in (a).

problem # 12.1 continues

(d) Energy from the battery: $\int_0^\infty V_0 I \, dt = \frac{V_0^2}{R}\int_0^\infty e^{-t/RC} dt =$

$$= \frac{V_0^2}{R} RC \left[-e^{-t/RC} \right]_0^\infty = \boxed{V_0^2 C}$$

since the current in (c) is the same as in (a), also the energy delivered to the resistor is the same $\boxed{\frac{1}{2} C V_0^2}$

The final energy in the capacitor is again $\boxed{\frac{1}{2} C V_0^2}$.

Half the energy from the battery goes to the capacitor and the other half to the resistor.

Solution of problem # 12.2

(a) $I = \int \vec{J} \cdot d\vec{a}$, where the integral is taken over a surface enclosing the positively charged conductor, e.g. "1". But $\vec{J} = \sigma \vec{E}$, and with Gauss's law $\int \vec{E} \cdot d\vec{a} = \frac{Q}{\varepsilon_0}$, we have

$$I = \sigma \int \vec{E} \cdot d\vec{a} = \frac{\sigma}{\varepsilon_0} Q .$$

Now $Q = CV$ and $V = IR$, so that $I = \frac{\sigma}{\varepsilon_0} CRI$ or $\boxed{R = \frac{\varepsilon_0}{\sigma C}}$

(b) $Q = CV = CRI \Rightarrow \frac{dQ}{dt} = -I = -\frac{1}{CR} Q$,

$Q(t) = Q_0 e^{-t/RC}$ or since $V = \frac{Q}{C}$, $V(t) = V_0 e^{-t/RC}$.

the time constant is $\boxed{\tau = RC = \frac{\varepsilon_0}{\sigma}}$ (from (a)).

Solution of problem # 12.3

(a) $\mathcal{E} = -\dfrac{d\Phi}{dt} = -Bl\dfrac{dx}{dt} = -Blv$

x is the distance between resistor and bar

$\mathcal{E} = RI \quad \Rightarrow \quad \boxed{I = \dfrac{Blv}{R}}$,

$\vec{v} \times \vec{B}$ is upward, so the current is downward through the resistor

(b) $\boxed{F = IlB = \dfrac{B^2 l^2 v}{R}}$ to the left (opposite to $\vec{v}$)

(c) $F = ma = m\dfrac{dv}{dt} = -\dfrac{B^2 l^2 v}{R} \quad \Rightarrow \quad \dfrac{dv}{dt} = -\dfrac{B^2 l^2}{mR} v$

$$\boxed{v = v_0 \exp\left[-\frac{B^2 l^2}{mR} t\right]}$$

(d) Initial KE of bar is $\frac{1}{2} m v_0^2$. The energy goes into the resistor.
The power delivered to the resistor is $I^2 R$, so

$$\frac{dW}{dt} = I^2 R = \frac{B^2 l^2 v^2}{R} = \frac{B^2 l^2}{R} v_0^2 \exp\left[-2\frac{B^2 l^2}{mR} t\right]$$

The total energy delivered to the resistor is

$$W = \frac{B^2 l^2}{R} v_0^2 \int_0^\infty dt \exp\left[-2\frac{B^2 l^2}{mR} t\right] = \frac{m v_0^2}{2}\left[-\exp\left[-2\frac{B^2 l^2}{mR} t\right]\right]_0^\infty = \frac{m v_0^2}{2}$$

✓

Solution of problem #12.4

(a) The field of the long wire is $\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$

$$\boxed{\Phi = \int \vec{B}\cdot d\vec{a} = \frac{\mu_0 I}{2\pi}\int_s^{s+a}\frac{1}{s}(a\,ds) = \frac{\mu_0 I}{2\pi}a\ln\frac{s+a}{s}}$$

(b) $$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{\mu_0 I}{2\pi}a\frac{d}{dt}\ln\frac{s+a}{s} \qquad \frac{ds}{dt} = v$$

hence $$\boxed{\mathcal{E} = -\frac{\mu_0 I}{2\pi}a\left[\frac{1}{s+a}-\frac{1}{s}\right]v = \frac{\mu_0 I}{2\pi}a^2 v\frac{1}{s(s+a)}}$$

The field points out of the page, so the force on a charge in the nearby side of the square is to the right ($\vec{F}\propto\vec{v}\times\vec{B}$). In the far side it is also to the right, however the field is weaker. Hence, the current flows <u>counterclockwise</u>.

(c) If the loop is pulled to the right at <u>speed v</u>, the flux does not change with time. If the flux is constant : $\boxed{\mathcal{E} = 0}$

Solution of problem # 12.5

$$\vec{B}(t) = B_0 \cos(\omega t)\,\hat{z}$$

$$\Phi = \pi \left(\frac{a}{2}\right)^2 B = \frac{\pi a^2}{4} B_0 \cos(\omega t)$$

$$\mathcal{E} = -\frac{d\Phi}{dt} = \frac{\pi a^2}{4} B_0 \omega \sin(\omega t)$$

$$\boxed{I(t) = \frac{\mathcal{E}}{R} = \frac{\pi a^2 \omega}{4R} B_0 \sin(\omega t)}$$