Solution of problem # 14.1

(a) $B = \mu_0 n I \Rightarrow \Phi_1 = \mu_0 n I \pi R^2$ (flux through a single turn)

In a length l there are nl turns. The total flux is

$\Phi = \mu_0 n^2 \pi R^2 I l = L l I$,

$\boxed{L = \mu_0 n^2 \pi R^2}$ self-inductance per unit length.

(b) $W = \frac{1}{2} L l I^2 = \boxed{\frac{1}{2}\mu_0 n^2 \pi R^2 l I^2}$

(c) $W = \frac{1}{2}\oint (\vec{A}\cdot\vec{I})\, dl$, $\vec{A} = (\mu_0 n I/2) s \hat{\phi}$ for $s \leq R$ from example

$\vec{B} = \vec{\nabla}\times\vec{A} = -\frac{\partial A_\phi}{\partial z}\hat{s} + \frac{1}{s}\frac{\partial (s A_\phi)}{\partial s}\hat{z} = \mu_0 n I \hat{z}$ ✓

For one turn: $W_1 = \frac{1}{2}\frac{\mu_0 n I}{2} R\,(I 2\pi R)$

For nl turns: $W = \frac{\mu_0 \pi n^2 l R^2}{2} I^2$

(d) $W = \frac{1}{2\mu_0}\int B^2 d\tau$ field is zero outside solenoid

$= \frac{1}{2\mu_0}(\mu_0 n I)^2 \int_{\text{cylinder of length } l} d\tau = \frac{1}{2\mu_0}(\mu_0 n I)^2 \pi R^2 l =$

$= \frac{\mu_0 n^2 \pi R^2 l}{2} I^2$

Solution of problem # 14.2

(a) The magnetic field of a solenoid is

$$\vec{B} = \begin{cases} \mu_0 K\hat{z} = \mu_0 \sigma \omega R \hat{z} & , s<R \\ 0 & , s>R . \end{cases} \qquad (K = \sigma v = \sigma \omega R)$$

From symmetry $\vec{E} \parallel \hat{\phi}$;

$$\oint \vec{E}\cdot d\vec{l} = E(2\pi s) = -\frac{d\Phi}{dt} = -\frac{d}{dt}(\pi s^2 B) = \begin{cases} -\pi s^2 \frac{dB}{dt} & \text{if } s<R \\ -\pi R^2 \frac{dB}{dt} & \text{if } s>R \end{cases}$$

$$\vec{E} = -\frac{s}{2}\frac{dB}{dt}\hat{\phi} \quad (s<R) \ ; \ \vec{E} = -\frac{R^2}{2s}\frac{dB}{dt}\hat{\phi} \quad (s>R) .$$

$$\vec{E} = \begin{cases} -\frac{s}{2}\frac{dB}{dt}\hat{\phi} = -\frac{sR}{2}\mu_0 \sigma \dot{\omega} \hat{\phi} & , s<R \\ -\frac{R^2}{2s}\frac{dB}{dt}\hat{\phi} = -\frac{R^3}{2s}\mu_0 \sigma \dot{\omega}\hat{\phi} & , s>R . \end{cases}$$

At the surface $(s=R)$ we have $\vec{E} = -\frac{1}{2}\mu_0 R^2 \sigma \dot{\omega} \hat{\phi}$.

The torque on a length l of the cylinder is

$$\vec{N} = \vec{r}\times\vec{F} = -R\underbrace{(\sigma 2\pi R l)}_{\text{charge}}\left(\frac{1}{2}\mu_0 R^2 \sigma \dot{\omega}\right)\underbrace{(\hat{s}\times\hat{\phi})}_{\hat{z}} = -\pi \mu_0 \sigma^2 R^4 \dot{\omega} l \hat{z}$$

The work done by the field per unit length is $W = \int \vec{N}\cdot(d\phi \hat{z})$

$$\frac{W}{l} = -\pi \mu_0 \sigma^2 R^4 \int (\dot{\omega}\hat{z})(d\phi \hat{z}) = -\pi\mu_0 \sigma^2 R^4 \int_0^{\omega_f} d\omega \overbrace{\frac{d\phi}{dt}}^{=\omega} =$$

$$= -\pi \mu_0 \sigma^2 R^4 \frac{\omega_f^2}{2} = \boxed{-\frac{\mu_0 \pi}{2}(\sigma R^2 \omega_f)^2}$$

This is the work done by the electric field. The work you must exert is

$$\boxed{\frac{W}{l} = \frac{\mu_0 \pi}{2}(\sigma R^2 \omega_f)^2} \quad \text{(without the minus sign).}$$

problem # 14.2 continues

(b) $W = \frac{1}{2\mu_0} \int_{\text{all space}} B^2 d\tau = \frac{1}{2}\mu_0 K^2 \cdot \pi R^2 l$ because the field is uniform inside the solenoid and zero outside, $K = \sigma \omega_f R$, so

$$\boxed{\frac{W}{l} = \frac{\pi}{2}\mu_0 \left(\sigma \omega_f R^2\right)^2}$$

Solution of problem # 14.3

(a) The displacement current is $\vec{J}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

Parallel plate capacitor $\quad \vec{E} = \frac{\sigma}{\varepsilon_0}\hat{z} = \frac{1}{\varepsilon_0}\frac{Q}{A}\hat{z}$

$$\vec{J}_d = \frac{1}{A}\frac{dQ}{dt}\hat{z} = \frac{I}{A}\hat{z} = \frac{I}{\pi a^2}\hat{z}$$

Amperian loop of radius s (between plates): $\vec{B} \propto \hat{\phi}$

$$\oint \vec{B}\cdot d\vec{l} = B\,2\pi s = \mu_0 I_{denc} = \mu_0 \frac{I}{\pi a^2}\pi s^2 = \mu_0 I \left(\frac{s}{a}\right)^2$$

$$B = \frac{\mu_0 I s^2}{2\pi s a^2}, \quad \boxed{\vec{B} = \frac{\mu_0 I s}{2\pi a^2}\hat{\phi}}$$

(b) (i) $\vec{E} = \frac{\sigma(t)}{\varepsilon_0}\hat{z}$, $\quad \sigma(t) = \frac{Q(t)}{\pi a^2} = \frac{It}{\pi a^2} \quad (I = \text{const})$

$$\boxed{\vec{E} = \frac{It}{\pi a^2 \varepsilon_0}\hat{z}}$$

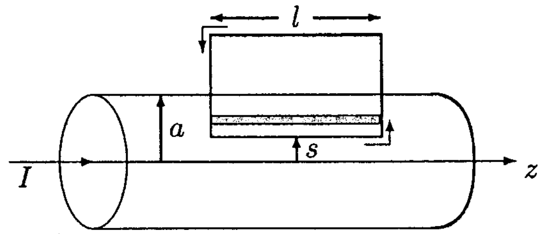
(ii) $I_{denc} = J_d \pi s^2 = \varepsilon_0 \frac{\partial E}{\partial t}\pi s^2 = \cancel{\varepsilon_0}\cancel{\pi} s^2 \frac{I}{\cancel{\pi} a^2 \cancel{\varepsilon_0}} = I\left(\frac{s}{a}\right)^2$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{denc} = B\,2\pi s \qquad \boxed{\vec{B} = \frac{\mu_0 I}{2\pi a^2} s\hat{\phi}}$$

Solution of problem # 14.4

(a) In the quasistatic approximation the magnetic field is "circumferential". Hence, the electric field is longitudinal (orthogonal to the magnetic field and by symmetry).

(b) Use the amperian loop shown in the figure. Outside the coaxial tube, $\vec{B}=0$ and $\vec{E}=0$.



Hence, $\oint \vec{E}\cdot d\vec{l} = El = -\frac{d\Phi}{dt} =$

$$= -\frac{d}{dt}\int \vec{B}\cdot d\vec{a} = -\frac{d}{dt}\int_s^a \frac{\mu_0 I}{2\pi s'}\, l\, ds'$$

So $\vec{E} = -\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln\left(\frac{a}{s}\right)\hat{z}$

$I = I_0\cos(\omega t)$, $\frac{dI}{dt} = -\omega I_0 \sin(\omega t)$, $\boxed{\vec{E} = \frac{\mu_0 I_0 \omega}{2\pi}\sin(\omega t)\ln\left(\frac{a}{s}\right)\hat{z}}$

(c) $\vec{J}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \varepsilon_0 \frac{\mu_0 I \omega^2}{2\pi}\cos(\omega t)\ln\left(\frac{a}{s}\right)\hat{z} = \frac{\mu_0\varepsilon_0}{2\pi}\omega^2 I \ln\left(\frac{a}{s}\right)\hat{z}$

(d) The displacement current:

$$I_d = \int \vec{J}_d\cdot d\vec{a} = \frac{\mu_0\varepsilon_0}{2\pi}\omega^2 I\int_0^a \ln\left(\frac{a}{s}\right)(2\pi s\, ds) = \frac{\mu_0\varepsilon_0}{\cancel{2\pi}}\omega^2 I \cancel{2\pi}\int_0^a (s\ln a - s\ln s)\,ds =$$

$$= \mu_0\varepsilon_0\omega^2 I\left[\ln a\,\frac{s^2}{2} - \frac{s^2}{2}\ln s + \frac{s^2}{4}\right]_0^a = \boxed{\frac{\mu_0\varepsilon_0\omega^2 I a^2}{4}}$$

(e) $\frac{I_d}{I} = \frac{\mu_0\varepsilon_0\omega^2 a^2}{4} = \left(\frac{\omega a}{2c}\right)^2$, $a = 10^{-3}\,m$, if $\frac{I_d}{I} = 0.01$ we have

$\frac{\omega a}{2c} = \frac{1}{10}$, $\omega = \frac{2c}{10a} = \frac{3\times 10^8\,\frac{m}{s}}{5\times 10^{-3}\,m} = 0.6\times 10^{11}\,\frac{1}{s} = 6\times 10^{10}\,\frac{1}{s}$

problem # 14.4 continues

$$\nu = \frac{\omega}{2\pi} \simeq 10^{10}\ \text{Hz} \quad \text{or} \quad 10\,\text{GHz}$$

this is a microwave frequency, way above radio frequencies.

Solution of problem # 14.5

(a) Assume that magnetic monopoles exist. Gauss's and Ampère's laws are not expected to change.

Obviously, the divergence of $\vec{B}$ is no longer zero, but

$\boxed{\vec{\nabla}\cdot\vec{B} = \alpha_0 \rho_m}$, where ρ_m is the density of magnetic charge, and α_0 is a constant (similar to ε_0 and μ_0).

The curl of $\vec{E}$ becomes $\boxed{\vec{\nabla}\times\vec{E} = \beta_0 \vec{J}_m - \frac{\partial \vec{B}}{\partial t}}$,

where $\vec{J}_m$ is the magnetic current density (flow of magnetic charge), and β_0 is another constant.

Assuming that the magnetic charge is conserved, then ρ_m and $\vec{J}_m$ satisfy the continuity equation:

$\boxed{\vec{\nabla}\cdot\vec{J}_m + \frac{\partial \rho_m}{\partial t} = 0}$.

Here $\beta_0 = -\alpha_0$.

We may choose $\alpha_0 = \mu_0$

The force law (Lorentz force for magnetic charges), should be something of the form: $q_m\left[\vec{B} + \gamma_0(\vec{v}\times\vec{E})\right]$, where γ_0 needs to have the dimension of an inverse velocity squared (E has the same dimension as vB). A natural candidate for γ_0 is $\frac{1}{c^2} = \mu_0\varepsilon_0$, so that

$$\boxed{\vec{F} = q_e\left[\vec{E} + (\vec{v}\times\vec{B})\right] + q_m\left[\vec{B} - \frac{1}{c^2}(\vec{v}\times\vec{E})\right]}$$

(the minus sign comes in because Faraday's law has $-\frac{\partial \vec{B}}{\partial t}$, while Maxwell's correction to Ampère's law is $+\mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}$).

problem # 14.5 continues

Now the magnetic analog to Coulomb's law reads

$$\vec{F} = \frac{\alpha_0}{4\pi} \frac{q_{m_1} q_{m_2}}{r^2} \hat{r} .$$

To determine α_0 we first introduce an arbitrary unit of magnetic charge and then measure the force between unit charges at a given separation.

(b) From $\vec{\nabla} \cdot \vec{B} = \mu_0 \rho_m$ it follows that the field of a point monopole is $\vec{B} = \frac{\mu_0}{4\pi} \frac{q_m}{r^2} \hat{r}$. The force has the form $\vec{F} \propto q_m \left(\vec{B} - \frac{1}{c^2} \vec{v} \times \vec{E}\right)$.

The proportionality constant must be 1 to reproduce "Coulomb's law" for magnetic point charges at rest. Hence,

$$\boxed{\vec{F} = q_m \left(\vec{B} - \frac{1}{c^2} \vec{v} \times \vec{E}\right)} .$$