Solution of problem # 15.1

(a) potential difference V $\Rightarrow$ cables are charged: λ

$$\left.\begin{aligned}\vec{E} &= \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{s}\hat{s}\\ \vec{B} &= \frac{\mu_0 I}{2\pi}\frac{1}{s}\hat{\phi}\end{aligned}\right\} \quad \vec{S} = \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right) = \frac{\lambda I}{4\pi^2\varepsilon_0}\frac{1}{s^2}\hat{z}$$

Power: $$P = \int \vec{S}\cdot d\vec{a} = \int_a^b S(2\pi s)\,ds = \frac{\lambda I}{2\pi\varepsilon_0}\int_a^b \frac{ds}{s} = \frac{\lambda I}{2\pi\varepsilon}\ln\left(\frac{b}{a}\right)$$

potential: $$V = \int_a^b \vec{E}\cdot d\vec{l} = \frac{\lambda}{2\pi\varepsilon_0}\int_a^b \frac{ds}{s} = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right)$$

Hence, $\boxed{P = VI}$

(b) charging capacitor of problem # 14.3

(i) $\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{z}$, $\sigma = \frac{Q}{A} = \frac{Q}{\pi a^2}$, $Q(t) = It$ $\Rightarrow$ $\boxed{\vec{E}(t) = \frac{It}{\pi\varepsilon_0 a^2}\hat{z}}$

$$\int(\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \oint \vec{B}\cdot d\vec{l} = B\,2\pi s = \mu_0\varepsilon_0\int\frac{\partial\vec{E}}{\partial t}\cdot d\vec{a} = \mu_0\varepsilon_0\frac{I}{\pi\varepsilon_0 a^2}\pi s^2$$

$$\boxed{\vec{B}(s,t) = \frac{\mu_0 I s}{2\pi a^2}\hat{\phi}}$$

(ii) $$u_{em} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) = \frac{1}{2}\left[\varepsilon_0\left(\frac{It}{\pi\varepsilon_0 a^2}\right)^2 + \frac{1}{\mu_0}\left(\frac{\mu_0 I s}{2\pi a^2}\right)^2\right] =$$

$$= \frac{1}{2}\frac{I^2}{\pi^2 a^4}\left[\frac{t^2}{\varepsilon_0} + \mu_0\left(\frac{s}{2}\right)^2\right] = \boxed{\frac{\mu_0 I^2}{2\pi^2 a^4}\left[(ct)^2 + \left(\frac{s}{2}\right)^2\right]}$$

$$\vec{S} = \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right) = \frac{1}{\mu_0}\left(\frac{It}{\pi\varepsilon_0 a^2}\cdot\frac{\mu_0 I s}{2\pi a^2}\right)\underbrace{(\hat{z}\times\hat{\phi})}_{-\hat{s}} = \boxed{-\frac{I^2 t}{2\pi^2\varepsilon_0 a^4}s\,\hat{s}}$$

$$\frac{\partial u_{em}}{\partial t} = \frac{\mu_0 I^2}{\pi^2 a^4}c^2 t = \frac{I^2 t}{\pi^2 a^4 \varepsilon_0}$$

$$-\vec{\nabla}\cdot\vec{S} = \frac{I^2 t}{2\pi^2\varepsilon_0 a^4}\vec{\nabla}\cdot(s\hat{s}) = \frac{I^2 t}{2\pi^2\varepsilon_0 a^4}\frac{1}{s}\frac{d}{ds}s^2 = \frac{I^2 t}{\pi^2\varepsilon_0 a^4} = \frac{\partial u_{em}}{\partial t}$$ ✓

problem # 15.1 continues

(iii) $\boxed{U_{em}} = \int u_{em}\, w\, 2\pi s\, ds =$ $\qquad w$ = width of gap

$$= 2\pi w \frac{\mu_0 I^2}{2\pi^2 a^4} \int_0^a \left[(ct)^2 + \left(\frac{s}{2}\right)^2\right] s\, ds = \frac{\mu_0 w I^2}{\pi a^4}\left[(ct)^2 \frac{a^2}{2} + \frac{a^4}{16}\right] =$$

$$= \boxed{\frac{\mu_0 w I^2}{2\pi a^2}\left[(ct)^2 + \frac{a^2}{8}\right]}$$

$$\boxed{P_{in}} = -\int \vec{S}\cdot d\vec{a} = \frac{I^2 t}{2\pi^2 \varepsilon_0 a^4}\,(a\hat{s})\cdot(2\pi a w \hat{s}) = \boxed{\frac{I^2 t w}{\pi \varepsilon_0 a^2}}$$

$$\frac{dU_{em}}{dt} = \frac{\mu_0 w I^2}{\pi a^2} c^2 t = \frac{w I^2}{\pi \varepsilon_0 a^2} t = P_{in}$$ ✓

Solution of problem #15.2

(a) The force is along the z-direction

$$(\overleftrightarrow{T}\cdot d\vec{a})_z = T_{zx}\,da_x + T_{zy}\,da_y + T_{zz}\,da_z \quad . \quad d\vec{a} \parallel \hat{z}$$

The plane has no area in the x and y directions: $da_x = da_y = 0$;
$da_z = -r\,dr\,d\phi$ if we calculate the force on the upper charge

$$(\overleftrightarrow{T}\cdot d\vec{a})_z = \varepsilon_0\left(E_zE_z - \frac{1}{2}E^2\right)(-r\,dr\,d\phi)$$

The field is $\vec{E} = \frac{1}{4\pi\varepsilon_0}\,2\frac{q}{\rho^2}\cos\theta\,\hat{r}$ (cylindrical coordinates)

$$\cos\theta = \frac{r}{\rho}, \quad \rho = \sqrt{r^2+a^2}\;; \quad E_z = 0, \quad E^2 = \left(\frac{q}{2\pi\varepsilon_0}\right)^2\frac{r^2}{(r^2+a^2)^3}$$

Therefore

$$\boxed{F_z = \frac{1}{2}\varepsilon_0\left(\frac{q}{2\pi\varepsilon_0}\right)^2 2\pi\int_0^\infty dr\,\frac{r^3}{(r^2+a^2)^3}} = \frac{q^2}{4\pi\varepsilon_0}\,\frac{1}{2}\int_0^\infty\frac{u\,du}{(u+a^2)^3} =$$

where $u = r^2$,

$$= \frac{q^2}{4\pi\varepsilon_0}\,\frac{1}{2}\left[-\frac{1}{u+a^2} + \frac{a^2}{2(u+a^2)^2}\right]_0^\infty = \frac{q^2}{4\pi\varepsilon_0}\,\frac{1}{2}\left[\frac{1}{a^2} - \frac{1}{2}\frac{1}{a^2}\right] =$$

$$\boxed{= \frac{q^2}{4\pi\varepsilon_0}\,\frac{1}{(2a)^2}} \quad \checkmark$$

(b) If the charges have opposite signs:

$$\vec{E} = -\frac{1}{4\pi\varepsilon_0}\,2\frac{q}{\rho^2}\sin\theta\,\hat{z}, \quad \sin\theta = \frac{a}{\rho}, \quad da_z = -r\,dr\,d\phi$$

$$E_z^2 = E^2 = \left(\frac{qa}{2\pi\varepsilon_0}\right)^2\frac{1}{(r^2+a^2)^3}, \quad (\overleftrightarrow{T}\cdot d\vec{a})_z = -\frac{\varepsilon_0}{2}\left(\frac{qa}{2\pi\varepsilon_0}\right)^2\frac{r\,dr\,d\phi}{(r^2+a^2)^3}, \text{ and}$$

$$\boxed{F_z = -\frac{\varepsilon_0}{2}\left(\frac{qa}{2\pi\varepsilon_0}\right)^2 2\pi\int_0^\infty\frac{r\,dr}{(r^2+a^2)^3} = -\frac{q^2a^2}{4\pi\varepsilon_0}\left[-\frac{1}{4}\frac{1}{(r^2+a^2)^2}\right]_0^\infty = -\frac{q^2}{4\pi\varepsilon_0}\,\frac{1}{(2a)^2}} \quad \checkmark$$

Solution of problem # 15.3

(a) $\vec{g}_{em} = \mu_0 \varepsilon_0 \vec{S} = \varepsilon_0 (\vec{E} \times \vec{B})$ momentum density

$\vec{E} = E\hat{z}$, $\vec{B} = B\hat{x}$, $g_{em} = \varepsilon_0 E B (\hat{z} \times \hat{x}) = \varepsilon_0 E B \hat{y}$

momentum:

$$\boxed{\vec{P}_{em} = \vec{g}_{em} \times \text{Volume} = \varepsilon_0 E B A d\,\hat{y}}$$

(b) impulse: $\vec{I}_{mp} = \int_0^\infty \vec{F} \cdot dt$; $\vec{F} = I\vec{l} \times \vec{B} = IBd(\hat{z} \times \hat{x}) = IBd\,\hat{y}$

$I_{mp} = \int_0^\infty IBd\, dt = -Bd \int_0^\infty dt \frac{dQ}{dt}$, $I = -\frac{dQ}{dt}$ (charge decreases)

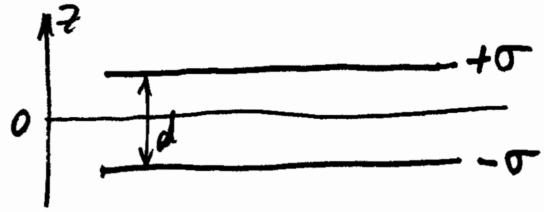
$$I_{mp} = -Bd\left[\underbrace{Q(t=\infty)}_{=0} - \underbrace{Q(t=0)}_{Q}\right] = BdQ$$

$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A}$ $Q = \varepsilon_0 A E$

$$\boxed{\vec{I}_{mp} = Bd\varepsilon_0 AE\hat{y} = Bd\varepsilon_0 EA\,\hat{y}}$$ equal to $\vec{P}_{em}$

Solution of problem # 15.4

(a) $E_x = E_y = 0$, $E_z = -\frac{\sigma}{\varepsilon_0}$



Hence, $T_{xy} = T_{xz} = T_{yz} = 0$

$$T_{xx} = T_{yy} = -\frac{\varepsilon_0}{2}E^2 = -\frac{\sigma^2}{2\varepsilon_0}$$

$$T_{zz} = \varepsilon_0\left(E_z^2 - \frac{1}{2}E^2\right) = \frac{\varepsilon_0}{2}E^2 = \frac{\sigma^2}{2\varepsilon_0}$$

$$\overleftrightarrow{T} = \frac{\sigma^2}{2\varepsilon_0}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $\vec{F} = \oint \overleftrightarrow{T}\cdot d\vec{a}$; note that $\vec{S} = 0$ since $\vec{B} = 0$.

For top plate : $d\vec{a} = -dx\,dy\,\hat{z}$ (outward into the capacitor)

By symmetry $\vec{F} \parallel \hat{z}$

$$F_z = \int T_{zz}\,da_z = -\frac{\sigma^2}{2\varepsilon_0}\int dx\int dy = -\frac{\sigma^2}{2\varepsilon_0}A$$

the force per unit area is $\boxed{\vec{f} = \frac{\vec{F}}{A} = -\frac{\sigma^2}{2\varepsilon_0}\hat{z}}$

(c) The momentum flux transported by the fields is $-\overleftrightarrow{T}\cdot d\vec{a}$;

per unit area in our case $-T_{zz} = \boxed{-\frac{\sigma^2}{2\varepsilon_0}}$; this is the momentum in the z-direction crossing a surface perpendicular to the z-axis, per unit area, per unit time.

(d) The recoil force is the momentum delivered, per unit time, so the force per unit area on the top plate is

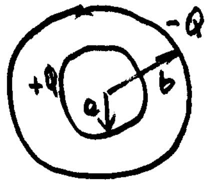
$$\boxed{\vec{f} = -\frac{\sigma^2}{2\varepsilon_0}\hat{z}}$$

problem # 15.4 continues

which is the same answer as in (b).

$$\left[\vec{F} = \frac{d\vec{p}_{mech}}{dt} = \underbrace{\cancel{-\mu_0 \epsilon_0 \frac{d}{dt}\int_{\mathcal{V}} \vec{S}\, d\tau}}_{=0} + \oint_S \overleftrightarrow{T} \cdot d\vec{a} = T_{zz}\, da\right]$$

Solution of problem # 15.5



$\uparrow \vec{B} = B_0 \hat{z}$

(a) Between the shells, $\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r}$

$$\vec{g} = \epsilon_0(\vec{E}\times\vec{B}) = \frac{QB_0}{4\pi}\frac{1}{r^2}(\hat{r}\times\hat{z})$$

$$\vec{L} = \int(\vec{r}\times\vec{g})\,d\tau = \frac{QB_0}{4\pi}\int\frac{1}{r^2}[\vec{r}\times(\hat{r}\times\hat{z})]\,r^2\sin\theta\,dr\,d\theta\,d\phi =$$

$$= \frac{QB_0}{4\pi}\int r[\hat{r}\times(\hat{r}\times\hat{z})]\sin\theta\,dr\,d\theta\,d\phi$$

$$\hat{r}\times(\hat{r}\times\hat{z}) = \hat{r}(\hat{r}\cdot\hat{z}) - \hat{z}(\hat{r}\cdot\hat{r}) = \hat{r}\cos\theta - \hat{z}$$

since $\vec{B}\parallel\hat{z}$, also $\vec{L}\parallel\hat{z}$; the z-component of the vector is

$$[\hat{r}\times(\hat{r}\times\hat{z})]_z = \cos^2\theta - 1 = -\sin^2\theta.$$

$$L_z = -\frac{QB_0}{4\pi}\int r\sin^3\theta\,dr\,d\theta\,d\phi = -\frac{QB_0}{4\pi}2\pi\int_0^\pi\sin^3\theta\,d\theta\int_a^b dr\,r =$$

$$= -\frac{QB_0}{2}\left(\frac{4}{3}\right)\left(\frac{b^2-a^2}{2}\right) = -\frac{1}{3}QB_0(b^2-a^2)$$

$$\boxed{\vec{L} = -\frac{1}{3}QB_0(b^2-a^2)\hat{z}}$$

(b) The changing magnetic field induces an electric field, which is obtained from Faraday's law. Amperian loop of radius s

$$\oint\vec{E}\cdot d\vec{l} = E(2\pi s) = -\frac{d\Phi}{dt} = -\frac{d}{dt}(\pi s^2 B(t)) = -\pi s^2\frac{dB}{dt};$$

therefore $\quad \vec{E} = -\frac{s}{2}\frac{dB}{dt}\hat{\phi}$.

The force on a surface patch da is: $d\vec{F} = \vec{E}\sigma\,da$, and the torque on this patch is $\quad d\vec{N} = \vec{s}\times d\vec{F}$. The net torque on the sphere of radius $\underline{a}$ is (we use $\vec{s} = a\sin\theta\,\hat{s}$ and $da = a^2\sin\theta\,d\theta\,d\phi$):

$$\vec{N}_a = \oint a\sin\theta\,\sigma\,a^2\sin\theta\left(-\frac{s}{2}\frac{dB}{dt}\right)d\theta\,d\phi\,\underbrace{(\hat{s}\times\hat{\phi})}_{\sin\theta\,\hat{z}} = -\frac{a^4}{2}\hat{z}\frac{Q}{4\pi a^2}\frac{dB}{dt}\int_0^{2\pi}d\phi\int_0^\pi\sin^3\theta\,d\theta$$

solution of problem # 15.5 continues

$$= -\frac{a^2}{4} Q \frac{dB}{dt}\hat{z}\left(\frac{4}{3}\right) = -\frac{Qa^2}{3}\frac{dB}{dt}\hat{z}.$$

Similarly, $\vec{N}_b = \frac{Qb^2}{3}\frac{dB}{dt}\hat{z}$, so the total torque is

$$\vec{N} = \frac{Q}{3}(b^2-a^2)\frac{dB}{dt}\hat{z}.$$

The angular momentum delivered to the spheres is

$$\boxed{\vec{L} = \int \vec{N}\,dt = \frac{Q}{3}(b^2-a^2)\hat{z}\int_{B_0}^{0}\frac{dB}{dt}dt = -\frac{Q}{3}(b^2-a^2)B_0\hat{z}}$$

The electromagnetic angular momentum has been entirely transformed into mechanical angular momentum.