Solution of problem # 16.1

$f(z,t) = A\sin(kz)\cos(kvt)$

$$\boxed{\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}}$$

wave equation

$$\frac{\partial^2 f}{\partial z^2} = -Ak^2 \sin(kz)\cos(kvt)$$

$$\frac{\partial^2 f}{\partial t^2} = -Ak^2v^2 \sin(kz)\cos(kvt) = v^2\frac{\partial^2 f}{\partial z^2} \quad \checkmark$$

satisfies wave equation

Use trig identity $\quad \sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$ to rewrite $f(z,t)$ as

$$\boxed{f(z,t) = \frac{A}{2}\left\{\sin[k(z+vt)] + \sin[k(z-vt)]\right\}}.$$

Two traveling waves, one moving to the left, the other to the right.

Solution of problem # 16.2

(a) $\boxed{\vec{k} = -\frac{\omega}{c}\hat{x} \quad , \quad \hat{n} = \hat{z}}$, $\vec{k}\cdot\vec{r} = -\frac{\omega}{c}x$, $\hat{k}\times\hat{n} = -\hat{x}\times\hat{z} = \hat{y}$

$$\boxed{\vec{E}(x,t) = E_0\cos\left(\frac{\omega}{c}x + \omega t\right)\hat{z} \ ; \ \vec{B}(x,t) = \frac{E_0}{c}\cos\left(\frac{\omega}{c}x + \omega t\right)\hat{y}}$$

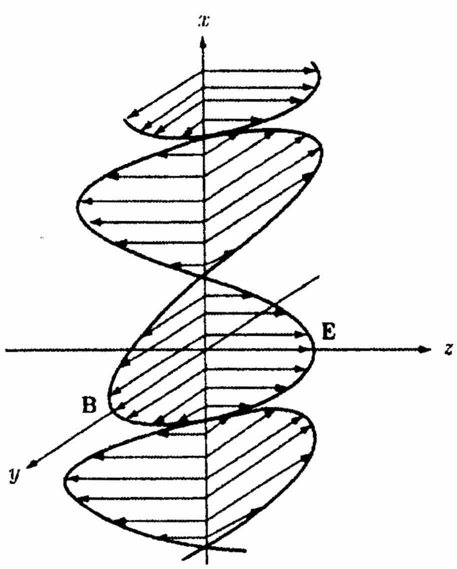
(b) $\boxed{\vec{k} = \frac{\omega}{c}\left(\frac{\hat{x}+\hat{y}+\hat{z}}{\sqrt{3}}\right) \quad , \quad \hat{n} = \frac{\hat{x}-\hat{z}}{\sqrt{2}}}$ since $\hat{n}$ is in the xz plane

and it has to be perpendicular to $\vec{k}$.

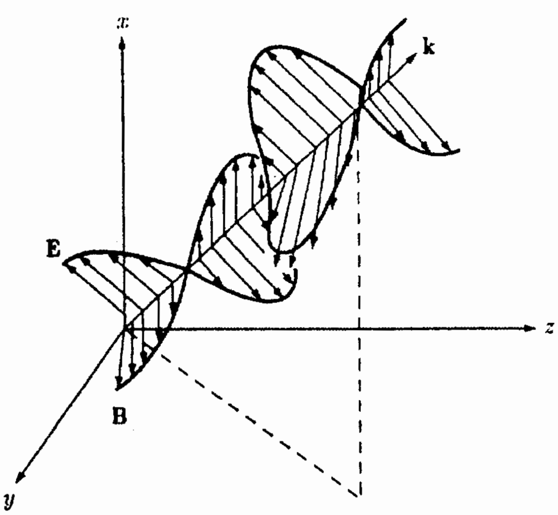
$$\vec{k}\cdot\vec{r} = \frac{\omega}{\sqrt{3}\,c}(\hat{x}+\hat{y}+\hat{z})\cdot(x\hat{x}+y\hat{y}+z\hat{z}) = \frac{\omega}{\sqrt{3}\,c}(x+y+z)$$

$$\hat{k}\times\hat{n} = \frac{1}{\sqrt{6}}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = \frac{1}{\sqrt{6}}(-\hat{x}+2\hat{y}-\hat{z})$$

$$\boxed{\begin{aligned} \vec{E}(x,y,z,t) &= E_0\cos\left[\frac{\omega}{\sqrt{3}\,c}(x+y+z) - \omega t\right]\left(\frac{\hat{x}-\hat{z}}{\sqrt{2}}\right) \\ \vec{B}(x,y,z,t) &= \frac{E_0}{c}\cos\left[\frac{\omega}{\sqrt{3}\,c}(x+y+z) - \omega t\right]\left(\frac{-\hat{x}+2\hat{y}-\hat{z}}{\sqrt{6}}\right) \end{aligned}}$$



(a)



(b)

Solution of problem # 16.3

Maxwell stress tensor:

$$T_{ij} = \varepsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)$$

$$\vec{E}(z,t) = E_0\cos(kz-\omega t)\,\hat{x} \quad , \quad \vec{B}(z,t) = \frac{1}{c}E_0\cos(kz-\omega t)\,\hat{y}$$

Hence, all off-diagonal matrix elements vanish. For the diagonal elements we have

$$T_{xx} = \varepsilon_0\left(E_xE_x - \frac{1}{2}E^2\right) + \frac{1}{\mu_0}\left(-\frac{1}{2}B^2\right) = \frac{1}{2}\left(\varepsilon_0E^2 - \frac{1}{\mu_0}B^2\right) = \frac{1}{2}\varepsilon_0\left(E^2 - E^2\right) = 0$$

$$T_{yy} = \varepsilon_0\left(-\frac{1}{2}E^2\right) + \frac{1}{\mu_0}\left(B_yB_y - \frac{1}{2}B^2\right) = \frac{1}{2}\left(-\varepsilon_0E^2 + \frac{1}{\mu_0}B^2\right) = -\frac{1}{2}\varepsilon_0\left(E^2 - E^2\right) = 0$$

$$T_{zz} = \varepsilon_0\left(-\frac{1}{2}E^2\right) + \frac{1}{\mu_0}\left(-\frac{1}{2}B^2\right) = -\frac{1}{2}\left(\varepsilon_0E^2 + \frac{1}{\mu_0}B^2\right) = -u$$

Hence, $\boxed{T_{zz} = -\varepsilon_0E_0^2\cos^2(kz-\omega t)}$ and all other elements are zero.

The momentum of the fields is in the z-direction, and it is being transported in the z direction. Hence, it makes sense that T_{zz} is the only non-zero element in T_{ij}.

$-\overleftrightarrow{T}\cdot d\vec{a}$ is the rate at which momentum crosses an area $d\vec{a}$. Here there is no momentum crossing areas oriented in the x or y direction. The momentum per unit time per unit area flowing across a surface oriented in the z direction is $-T_{zz} = u = gc$. Hence, $\Delta p = gcA\Delta t$ and consequently $\Delta p/\Delta t = gcA$ = momentum per unit time crossing an area A.

Hence, evidently $\boxed{\text{momentum flux density} = \text{energy density}}$

Solution of problem # 16.4

incoming fields:
$$\begin{cases} \tilde{\vec{E}}_I = \tilde{E}_{0I}\, e^{i(\vec{k}_I\cdot\vec{r}-\omega t)}\,\hat{y} \\ \tilde{\vec{B}}_I = \frac{1}{v_1}\tilde{E}_{0I}\, e^{i(\vec{k}_I\cdot\vec{r}-\omega t)}\left(-\cos\theta_1\,\hat{x}+\sin\theta_1\,\hat{z}\right) \end{cases}$$

reflected fields:
$$\begin{cases} \tilde{\vec{E}}_R = \tilde{E}_{0R}\, e^{i(\vec{k}_R\cdot\vec{r}-\omega t)}\,\hat{y} \\ \tilde{\vec{B}}_R = \tilde{E}_{0R}\frac{1}{v_1}\, e^{i(\vec{k}_R\cdot\vec{r}-\omega t)}\left(\cos\theta_1\,\hat{x}+\sin\theta_1\,\hat{z}\right) \end{cases}$$

transmitted fields:
$$\begin{cases} \tilde{\vec{E}}_T = \tilde{E}_{0T}\, e^{i(\vec{k}_T\cdot\vec{r}-\omega t)}\,\hat{y} \\ \tilde{\vec{B}}_T = \frac{1}{v_2}\tilde{E}_{0T}\, e^{i(\vec{k}_T\cdot\vec{r}-\omega t)}\left(-\cos\theta_2\,\hat{x}+\sin\theta_2\,\hat{z}\right) \end{cases}$$

(a) Boundary conditions: for $z=0$, $k_I\cdot\vec{r}-\omega t = k_R\cdot\vec{r}-\omega t = k_T\cdot\vec{r}-\omega t$, so that we can drop all exponential factors in applying the boundary conditions.

Law of refraction: $\dfrac{\sin\theta_2}{\sin\theta_1} = \dfrac{v_2}{v_1}$

(i) $0=0$ (trivial, because $\vec{E}\perp\hat{z}$)

(iii) $\tilde{\vec{E}}_{0I} + \tilde{\vec{E}}_{0R} = \tilde{\vec{E}}_{0T}$ (since $\vec{E}\parallel\hat{y}$), $\boxed{\tilde{E}_{0I}+\tilde{E}_{0R}=\tilde{E}_{0T}}$

(ii) $\frac{1}{v_1}\tilde{E}_{0I}\sin\theta_1 + \frac{1}{v_1}\tilde{E}_{0R}\sin\theta_1 = \frac{1}{v_2}\tilde{E}_{0T}\sin\theta_2$ or $\tilde{E}_{0I}+\tilde{E}_{0R} =$
$= \dfrac{v_1\sin\theta_2}{v_2\sin\theta_1}\tilde{E}_{0T}$; by the law of refraction then $\tilde{E}_{0I}+\tilde{E}_{0R}=\tilde{E}_{0T}$
which is the same as (iii).

(iv) $\frac{1}{\mu_1}\left[\frac{1}{v_1}\tilde{E}_{0I}(-\cos\theta_1) + \frac{1}{v_1}\tilde{E}_{0R}\cos\theta_1\right] = \frac{1}{\mu_2 v_2}\tilde{E}_{0T}(-\cos\theta_2) \Rightarrow$

$$\tilde{E}_{0I} - \tilde{E}_{0R} = \frac{\mu_1 v_1}{\mu_2 v_2}\,\frac{\cos\theta_2}{\cos\theta_1}\,\tilde{E}_{0T}$$

define $\boxed{\alpha = \dfrac{\cos\theta_2}{\cos\theta_1}\ ,\ \beta = \dfrac{\mu_1 v_1}{\mu_2 v_2}}$ $\boxed{\tilde{E}_{0I}-\tilde{E}_{0R} = \alpha\beta\,\tilde{E}_{0T}}$

solution of 16.4 continues

We have to solve for $\tilde{E}_{0R}$ and $\tilde{E}_{0T}$

$$\tilde{E}_{0T} = \frac{2}{1+\alpha\beta}\tilde{E}_{0I} \quad , \quad \tilde{E}_{0R} = \frac{1-\alpha\beta}{1+\alpha\beta}\tilde{E}_{0I}$$

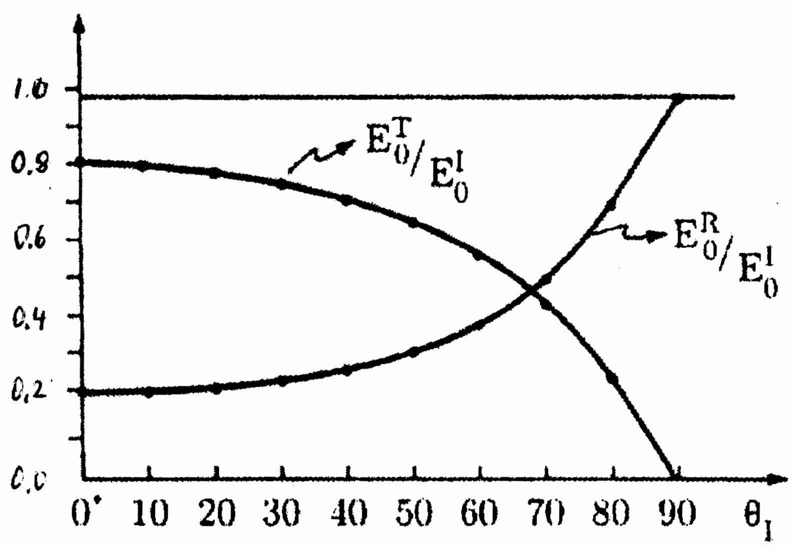
Since α and β are positive it follows that $\vec{\tilde{E}}_{0T}$ and $\vec{\tilde{E}}_{0I}$ are in phase and the real amplitudes are related by $\boxed{E_{0T} = \frac{2}{1+\alpha\beta}E_{0I}}$.

The reflected wave is in phase if $\alpha\beta<1$ and $180°$ out of phase if $\alpha\beta>1$. The real amplitudes are related by $\boxed{E_{0R} = \left|\frac{1-\alpha\beta}{1+\alpha\beta}\right|E_{0I}}$.

These are the Fresnel equations for polarization perpendicular to the plane of incidence.

(b) To construct the graphs, note that $\left(\beta \approx \frac{v_2}{v_1} = \frac{n_1}{n_2}\right)$

$$\alpha\beta = \beta\frac{\sqrt{1-\sin^2\theta_1/\beta^2}}{\cos\theta_1} = \frac{\sqrt{\beta^2-\sin^2\theta_1}}{\cos\theta_1} = \frac{\sqrt{2.25-\sin^2\theta_1}}{\cos\theta_1}$$



(c) $E_{0R}=0$ means $\alpha\beta=1$ or $\alpha = \frac{\sqrt{1-\left(\frac{v_2}{v_1}\right)^2\sin^2\theta}}{\cos\theta} = \frac{1}{\beta} = \frac{\mu_2 v_2}{\mu_1 v_1} \simeq \frac{v_2}{v_1}$

solution of problem # 16.4 continues

or $1-\left(\frac{v_2}{v_1}\right)^2 \sin^2\theta = \left(\frac{v_2}{v_1}\right)^2 \cos^2\theta$ or $1 = \left(\frac{v_2}{v_1}\right)^2$. This can only be true for optically indistinguishable media, in which case there is of course no reflection, but this would be true at <u>any</u> angle and not just at the Brewster's angle.

δ_R is either always 0 or always π for a given interface. δ_R cannot switch over as θ is changed, the way it does for polarization in the plane of incidence.

(d) At normal incidence, $\alpha = 1$, and Fresnel's equations reduce to

$E_{0T} = \frac{2}{1+\beta} E_{0I}$, $E_{0R} = \left|\frac{1-\beta}{1+\beta}\right| E_{0I}$, consistent with the results derived in class (note for normal incidence we cannot distinguish the cases of polarization parallel and perpendicular to the plane of incidence.

(e) Reflection and transmission coefficients:

$$\boxed{R = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2} \quad , \quad \boxed{T = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\alpha\left(\frac{E_{0T}}{E_{0I}}\right)^2 = \alpha\beta\left(\frac{2}{1+\alpha\beta}\right)^2}$$

$$R + T = \frac{1 - 2\alpha\beta + (\alpha\beta)^2 + 4\alpha\beta}{(1+\alpha\beta)^2} = 1 \quad \checkmark$$

Solution of problem #16.5

(a) Energy density of electromagnetic wave

$$u = \frac{1}{2}\left(\varepsilon E^2 + \frac{1}{\mu}B^2\right) = \frac{1}{2}e^{-2kz}\left[\varepsilon E_0^2 \cos^2(kz-\omega t+\delta_E) + \right.$$

$$\left. + \frac{1}{\mu}B_0^2 \cos^2(kz-\omega t+\delta_E+\phi)\right]$$

Recall that in a conducting medium the electric and magnetic fields are no longer in phase. Time-averaging over a full cycle:

$$\langle u \rangle = \frac{1}{2}e^{-2kz}\left[\frac{\varepsilon}{2}E_0^2 + \frac{1}{2\mu}B_0^2\right]$$

We know that the ratio of the amplitudes is

$$\frac{B_0}{E_0} = \frac{K}{\omega} = \sqrt{\varepsilon\mu\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}} \quad , \qquad K = |\tilde{k}|$$

so that

$$\langle u \rangle = \frac{1}{4}e^{-2kz}\varepsilon E_0^2\left[1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]$$

The real part of the complex wave number $\tilde{k}$ is

$$k = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right]^{1/2} \quad ; \quad 1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} = \frac{2}{\varepsilon\mu}\left(\frac{k}{\omega}\right)^2$$

and

$$\boxed{\langle u \rangle = \frac{1}{4}e^{-2kz}\varepsilon E_0^2 \frac{2}{\varepsilon\mu}\left(\frac{k}{\omega}\right)^2 = \frac{k^2}{2\mu\omega^2}E_0^2 e^{-2kz}}$$

Hence, the ratio of the magnetic contribution to the electric contribution is

$$\frac{\langle u_{mag}\rangle}{\langle u_{elec}\rangle} = \frac{B_0^2/\mu}{E_0^2\varepsilon} = \frac{1}{\varepsilon\mu}\varepsilon\mu\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} = \sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} > 1$$

(b) $$\vec{S} = \frac{1}{\mu}(\vec{E}\times\vec{B}) = \frac{1}{\mu}E_0B_0e^{-2kz}\cos(kz-\omega t+\delta_E)\cos(kz-\omega t+\delta_E+\phi)\hat{z}$$

problem #16.5 continues

$$\langle \vec{S} \rangle = \frac{1}{\mu} E_0 B_0 e^{-2kz} \langle \cos^2(kz-\omega t+\delta_E)\cos\phi - \cos(kz-\omega t+\delta_E) \times$$

$$\times \sin(kz-\omega t+\delta_E)\sin\phi \rangle \hat{z} = \frac{1}{2\mu} E_0 B_0 e^{-2kz} \cos\phi \, \hat{z}$$

Hence,

$$I = \frac{1}{2\mu} E_0 B_0 e^{-2kz} \cos\phi = \frac{1}{2\mu} E_0^2 e^{-2kz} \left(\frac{K}{\omega}\cos\phi\right),$$

but $k = K\cos\phi$, so that

$$\boxed{I = \frac{k}{2\mu\omega} E_0^2 e^{-2kz}}$$