Solution of problem #17.1

(a) The wave vector of an electromagnetic wave in a conductor is

$$\tilde{k} = k + i\kappa$$

$$k = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right]^{1/2}, \quad \kappa = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right]^{1/2}$$

if $\sigma \ll \omega\varepsilon$ $\quad \kappa \simeq \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\frac{1}{2}\left(\frac{\sigma}{\varepsilon\omega}\right)^2\right]^{1/2} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$

the skin depth is defined as $\boxed{d = \frac{1}{\kappa} \simeq \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}}$.

For pure water: $\varepsilon = 80.1\,\varepsilon_0$, $\mu \simeq \mu_0$, $\sigma = 0.4\times10^{-5}\,(\Omega\,m)^{-1}$

$$d = 2\times 2.5\,10^{5}\sqrt{\frac{80.1\cdot(8.85\times10^{-12})}{4\pi\times10^{-7}}}\,\Omega\,m\left[\frac{C^2}{N\,m^2}\cdot\frac{A^2}{N}\right]^{1/2} \quad ; \Omega = \frac{V}{A} = \frac{J}{A^2 s}$$

$$\boxed{d = 5\times10^{5}\sqrt{56.41\times10^{-5}}\,m = 11.88\times10^{3}\,m = 1.19\times10^{4}\,m}$$

(b) good conductor $\sigma \gg \omega\varepsilon$

$$k \simeq \kappa = \omega\sqrt{\frac{\varepsilon\mu}{2}}\sqrt{\frac{\sigma}{\varepsilon\omega}} = \sqrt{\frac{\omega\mu\sigma}{2}} \quad \text{and} \quad \lambda = \frac{2\pi}{k} \simeq \frac{2\pi}{\kappa} = 2\pi d, \quad \boxed{d = \frac{\lambda}{2\pi}}$$

$$\kappa \simeq \sqrt{\frac{\omega\mu\sigma}{2}} = \left(\frac{10^{15}s^{-1}\,4\pi\,\cancel{\times10^{-7}}\,N/A^2\,\cancel{\times10^{7}}}{2\,\Omega\,m}\right)^{1/2} \simeq 8\times10^{7}\left(\frac{A^2 s}{J\,m\,s}\,\frac{N}{A^2}\right)^{1/2} = 8\times10^{7}\,m^{-1}$$

$$\boxed{d = \frac{1}{\kappa} = \frac{1}{8}\times10^{-7}\,m \simeq 0.13\times10^{-7}\,m = 13\,nm}$$

The fields do not penetrate far into the metal → the metal is opaque

(c) good conductor, $k \simeq \kappa$, $\phi = \tan^{-1}\frac{\kappa}{k} \simeq \tan^{-1}1 \simeq 45°$ ✓

$$\frac{B_0}{E_0} = \frac{K}{\omega} = \sqrt{\varepsilon\mu\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}} \simeq \sqrt{\frac{\sigma\mu}{\omega}} \simeq \sqrt{\frac{10^{7}(4\pi\times10^{-7})}{10^{15}}\,\frac{s^2A^2}{J\,m}\,\frac{N}{A^2}} \simeq$$

$$\simeq 1.12\times10^{-7}\,\frac{s}{m}.$$

Solution of problem # 17.1 continues

Note that in vacuum the ratio is $1/c \simeq 3 \cdot 10^{-9}$ s/m. Hence, in a metal the magnetic field is about 100 times larger than in vacuum.

Solution of problem #17.2

For reflection at a conducting surface we have

$$\tilde{E}_{0R} = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right) E_{0I} \quad , \quad \tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega} \tilde{k}_2 = \frac{\mu_1 v_1}{\mu_2 \omega}(k_2 + i K_2)$$

so that

$$R = \left| \frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} \right|^2 = \left| \frac{1-\tilde{\beta}}{1+\tilde{\beta}} \right|^2 = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\left(\frac{1-\tilde{\beta}^*}{1+\tilde{\beta}^*}\right)$$

Since silver is a good conductor ($\sigma \gg \varepsilon\omega$)

$$k_2 = \omega\sqrt{\frac{\varepsilon_2\mu_2}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon_2\omega}\right)^2}+1\right]^{1/2} \simeq \omega\sqrt{\frac{\varepsilon_2\mu_2}{2}}\sqrt{\frac{\sigma}{\varepsilon_2\omega}} = \sqrt{\frac{\sigma\omega\mu_2}{2}} \simeq K_2$$

and $\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega}\sqrt{\frac{\sigma\omega\mu_2}{2}}(1+i) = \mu_1 v_1 \sqrt{\frac{\sigma}{2\mu_2\omega}}(1+i)$

Let $\gamma = \mu_1 v_1 \sqrt{\frac{\sigma}{2\mu_2\omega}} = \mu_0 c\sqrt{\frac{\sigma}{2\mu_0\omega}} = c\sqrt{\frac{\mu_0\sigma}{2\omega}} = \left(3\times 10^8 \frac{m}{s}\right)\times\left(\frac{4\pi 10^{-7}\times 6\times 10^7}{2\times 4\times 10^{15}}\right)^{1/2}$

$$\times\left(\frac{s}{\Omega m}\cdot\frac{N}{A^2}\right)^{1/2} = 29\frac{m}{s}\left(\frac{Ns}{\frac{V}{A}m A^2}\right)^{1/2} = 29\frac{m}{s}\left(\frac{Ns}{Jm}\right)^{1/2} = 29\frac{m}{s}\left(\frac{Ns^2}{Nm^2}\right)^{1/2} = 29 \checkmark$$

Then

$$R = \left(\frac{1-\gamma-i\gamma}{1+\gamma+i\gamma}\right)\left(\frac{1-\gamma+i\gamma}{1+\gamma-i\gamma}\right) = \frac{(1-\gamma)^2+\gamma^2}{(1+\gamma)^2+\gamma^2} = 0{,}933$$

93% of the light is reflected (good mirror)

Solution of problem # 17.3

For TEM modes both, $E_z = 0$ and $B_z = 0$.

If $E_z = 0$, Gauss's law, $\vec{\nabla} \cdot \vec{E} = 0$, yields

$$\boxed{\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0} \qquad \text{divergence of } \tilde{\vec{E}}_0 = 0 \,;$$

if $B_z = 0$, Faraday's law, $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, yields

$$\boxed{\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0} \qquad \text{rotator of } \tilde{\vec{E}}_0 = 0 .$$

Since $\tilde{\vec{E}}_0$ has zero divergence and zero curl, it can be written as the gradient of a scalar potential that satisfies Laplace's equation. The boundary condition $\vec{E}^{\parallel} = 0$ requires that the surface be an equipotential. Since Laplace's equation admits no local maxima or minima, the potential is constant throughout. Hence, the electric field is <u>zero</u>.

Solution of problem # 17.4

From $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ and $\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$ we obtain

(i) $\frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x$ (ii) $ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y$

(iii) $\frac{\partial B_z}{\partial y} - ikB_y = -\frac{i\omega}{c^2}E_x$ (iv) $ikB_x - \frac{\partial B_z}{\partial x} = -i\frac{\omega}{c^2}E_y$

For TE modes $E_z = 0$. Their dispersion is $k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]}$

For $m = n = 0$ we have $k = \frac{\omega}{c}$.

From (i) $E_y = -cB_x$; from (ii) $E_x = cB_y$; from (iii)

$\frac{\partial B_z}{\partial y} = i\left(kB_y - \frac{\omega}{c^2}E_x\right) = ik\left(B_y - B_y\right) = 0$; and from (iv)

$\frac{\partial B_z}{\partial x} = i\left(kB_x + \frac{\omega}{c^2}E_y\right) = ik\left(B_x - B_x\right) = 0$.

Hence, since $\frac{\partial B_z}{\partial y} = \frac{\partial B_z}{\partial x} = 0$ and B_z does not depend on z, we have that $B_z = \text{const}$. Faraday's law in integral form says $\left(\vec{B} = \vec{B}_0 e^{i(kz-\omega t)}\right)$

$$\oint \vec{E}\cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a} = i\omega\int \vec{B}\cdot d\vec{a}$$

Applied to a cross section of the wave guide this yields

$\oint \vec{E}\cdot d\vec{l} = i\omega B_z e^{i(kz-\omega t)}ab$, where we could pull B_z out of the integral because it is constant. But if the boundary is just inside the metal, where $\vec{E} = 0$, it follows that $\boxed{B_z = 0}$. Hence, this corresponds to a TEM mode, which we already know cannot exist in this guide.

Solution of problem #17.5

For TM modes we have to solve ($B_z = 0$)

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right] E_z = 0$$

subject to the boundary condition $\vec{E}^{\parallel} = 0$.

Let $E_z(x,y) = X(x)Y(y)$, then we obtain

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2 \quad , \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = -k_y^2 \quad , \quad -k_x^2 - k_y^2 + \left(\frac{\omega}{c}\right)^2 - k^2 = 0$$

$$X(x) = A\sin(k_x x) + B\cos(k_x x) \quad , \quad Y(y) = C\sin(k_y y) + D\cos(k_y y) \, ,$$

the boundary condition requires that $E_z = 0$ when $x = 0$ and $x = a$, as well as $E_z = 0$ for $y = 0$ and $y = b$. Hence, $B = 0$ and $k_x = \frac{m\pi}{a}$, $m = 1, 2, \dots$ and $D = 0$ and $k_y = \frac{n\pi}{b}$, $n = 1, 2, 3, \dots$ Thus,

$$E_z = E_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \quad \text{with} \quad m, n = 1, 2, 3, \dots$$

Defining the cutoff frequency $\omega_{mn} = c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$, the wave velocity is $v = \frac{\omega}{k} = \frac{c}{\sqrt{1 - (\omega_{mn}/\omega)^2}}$, and the group velocity

$$v_g = \frac{1}{dk/d\omega} = c\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2} .$$

TM_{11} is the lowest mode with cutoff frequency $\omega_{11} = c\pi\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$.

The ratio of the lowest TM frequency to the lowest TE frequency (ω_{10}) is

$$\frac{c\pi\sqrt{1/a^2 + 1/b^2}}{c\pi/a} = \boxed{\sqrt{1 + \left(\frac{a}{b}\right)^2}}$$

The difference in the modes is the boundary condition: $B^{\perp} = 0$ for the TE and $\vec{E}^{\parallel} = 0$ for TM modes.