Solution of problem # 18.1

(a) $V(\vec{r},t)=0$, $\vec{A}(\vec{r},t) = -\frac{1}{4\pi\varepsilon_0}\frac{qt}{r^2}\hat{r}$

$$\boxed{\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}}$$

$$\boxed{\vec{B} = \vec{\nabla}\times\vec{A} = 0}$$

This corresponds to a stationary point charge q at the origin.
$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$ and $\vec{A}=0$ would be the more natural choice of potentials for this problem.
Evidently $\rho = q\,\delta^3(\vec{r})$ and $\vec{J}=0$

(b) $\lambda = -\frac{1}{4\pi\varepsilon_0}\frac{qt}{r}$;

$$\boxed{V' = V - \frac{\partial\lambda}{\partial t} = 0 - \left(-\frac{1}{4\pi\varepsilon_0}\frac{q}{r}\right) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}}$$

$$\boxed{\vec{A}' = \vec{A} + \vec{\nabla}\lambda = -\frac{1}{4\pi\varepsilon_0}\frac{qt}{r^2}\hat{r} + \left(-\frac{1}{4\pi\varepsilon_0}qt\right)\left(-\frac{1}{r^2}\hat{r}\right) = 0}$$

This gauge function transforms the problem into the "natural choice" of potentials mentioned in (a).

Solution of problem #18.2

$$\rho(\vec{r},t) = q(t)\,\delta^3(\vec{r}) \quad , \quad \vec{J}(\vec{r},t) = -\frac{1}{4\pi}\frac{\dot{q}}{r^2}\hat{r}$$

(a) Check continuity equation $\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0$

$$\vec{\nabla}\cdot\vec{J} = -\frac{\dot{q}}{4\pi}\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = -\frac{\dot{q}}{4\pi}4\pi\,\delta^3(\vec{r}) = -\dot{q}\,\delta^3(\vec{r}) = -\frac{\partial \rho}{\partial t} \quad \checkmark$$

(b) $$\boxed{V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,',t)}{\mathscr{r}}d\tau' = \frac{q(t)}{4\pi\varepsilon_0}\int\frac{\delta^3(\vec{r})}{\mathscr{r}}d\tau' = \frac{1}{4\pi\varepsilon_0}\frac{q(t)}{r}}$$

$\vec{\nabla}\cdot\vec{A} = 0$ Coulomb gauge , $\vec{B} = 0$ by symmetry

$\vec{\nabla}\times\vec{A} = \vec{B} = 0$, $\vec{A}\to 0$ for $r\to\infty$

Hence $\boxed{\vec{A} = 0}$

(c) Find the fields:

$$\boxed{\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}V = \frac{1}{4\pi\varepsilon_0}\frac{q(t)}{r^2}\hat{r}} \qquad \boxed{\vec{B} = 0}$$

Verify Maxwell's equations:

$$\vec{\nabla}\cdot\vec{E} = \frac{q(t)}{4\pi\varepsilon_0}\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = \frac{q(t)}{4\pi\varepsilon_0}4\pi\,\delta^3(\vec{r}) = \frac{q\,\delta^3(\vec{r})}{\varepsilon_0} = \frac{\rho}{\varepsilon_0} \quad \checkmark$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \checkmark$$

$$\vec{\nabla}\times\vec{E} = 0 = -\frac{\partial\vec{B}}{\partial t} \quad \checkmark$$

$$\vec{\nabla}\times\vec{B} = 0 \; ; \quad \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} = \mu_0\left(-\frac{1}{4\pi}\frac{\dot{q}}{r^2}\hat{r} + \varepsilon_0\frac{1}{4\pi\varepsilon_0}\frac{\dot{q}}{r^2}\hat{r}\right) = 0 \quad \checkmark$$

Solution of problem # 18.3

$V = 0$, $\vec{A} = A_0 \sin(kx - \omega t)\,\hat{y}$

$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} = -A_0 \cos(kx - \omega t)(-\omega)\,\hat{y} = \omega A_0 \cos(kx - \omega t)\,\hat{y}$

$\vec{B} = \vec{\nabla} \times \vec{A} = \hat{z}\frac{\partial}{\partial x}\left[A_0 \sin(kx - \omega t)\right] = A_0 k \cos(kx - \omega t)\,\hat{z}$

Hence, $\vec{\nabla} \cdot \vec{E} = 0$ ✓ , $\vec{\nabla} \cdot \vec{B} = 0$ ✓

$\vec{\nabla} \times \vec{E} = \hat{z}\frac{\partial}{\partial x}\left[\omega A_0 \cos(kx - \omega t)\right] = -k\omega A_0 \sin(kx - \omega t)\,\hat{z}$

$-\frac{\partial \vec{B}}{\partial t} = -A_0 k \omega \sin(kx - \omega t)\,\hat{z}$, so that $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ ✓

$\vec{\nabla} \times \vec{B} = -\hat{y}\frac{\partial}{\partial x}\left[A_0 k \cos(kx - \omega t)\right] = A_0 k^2 \sin(kx - \omega t)$

$\frac{\partial \vec{E}}{\partial t} = A_0 \omega^2 \sin(kx - \omega t)\,\hat{y}$, so that $\vec{\nabla} \times \vec{B} = \frac{k^2}{\omega^2}\frac{\partial \vec{E}}{\partial t}$.

Hence, if $\boxed{\omega = ck \ , \ c^2 = \frac{1}{\mu_0 \varepsilon_0}}$, $\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ ✓

Solution of problem # 18.4

$\vec{A} = -\frac{1}{2}(\vec{r}\times\vec{B})$ does not explicitly depend on t and $\vec{B}$ is independent of $\vec{r}$.

$$d\vec{A} = \left(\frac{\partial\vec{A}}{\partial x}\right)\frac{dx}{dt}dt + \left(\frac{\partial\vec{A}}{\partial y}\right)\frac{dy}{dt}dt + \left(\frac{\partial\vec{A}}{\partial z}\right)\frac{dz}{dt}dt + \left(\frac{\partial\vec{A}}{\partial t}\right)dt =$$

$$= \left[(\vec{v}\cdot\vec{\nabla})\vec{A} + \frac{\partial\vec{A}}{\partial t}\right]dt$$

$$\frac{d\vec{A}}{dt} = \frac{\partial\vec{A}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{A} = \vec{v}\cdot\vec{\nabla}\left(-\frac{1}{2}\vec{r}\times\vec{B}\right) =$$

$$= -\frac{1}{2}\left[v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right]\left[(yB_z - zB_y)\hat{x} + (zB_x - xB_z)\hat{y} + (xB_y - yB_x)\hat{z}\right] =$$

$$= -\frac{1}{2}\left[v_x(-B_z\hat{y} + B_y\hat{z}) + v_y(B_z\hat{x} - B_x\hat{z}) + (-B_y\hat{x} + B_x\hat{y})v_z\right] =$$

$$= -\frac{1}{2}\left[(v_yB_z - v_zB_y)\hat{x} + (v_zB_x - v_xB_z)\hat{y} + (v_xB_y - v_yB_x)\hat{z}\right] =$$

$$= -\frac{1}{2}(\vec{v}\times\vec{B}).$$ ✓

$$\frac{d}{dt}(\vec{p} + q\vec{A}) = \frac{d\vec{p}}{dt} - \frac{q}{2}(\vec{v}\times\vec{B}) = -\vec{\nabla}U_{vel} = -q\vec{\nabla}(\overset{=0}{V} - \vec{v}\cdot\vec{A}) =$$

$$= q\vec{\nabla}(\vec{v}\cdot\vec{A}) = -\frac{q}{2}\vec{\nabla}\left[\vec{v}\cdot(\vec{r}\times\vec{B})\right]$$ or

$$\frac{d\vec{p}}{dt} = \frac{q}{2}(\vec{v}\times\vec{B}) - \frac{q}{2}\vec{\nabla}\left[\vec{r}\cdot(\vec{B}\times\vec{v})\right]$$

Now, since $\vec{c} = \vec{B}\times\vec{v}$ is independent or $\vec{r}$, we have

$$\vec{\nabla}\left[\vec{r}\cdot(\vec{B}\times\vec{v})\right] = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left[xc_x + yc_y + zc_z\right] = c_x\hat{x} + c_y\hat{y} + c_z\hat{z} = \vec{c}$$

and $$\frac{d\vec{p}}{dt} = \frac{q}{2}(\vec{v}\times\vec{B}) - \frac{q}{2}(\vec{B}\times\vec{v}) = q(\vec{v}\times\vec{B}).$$ ✓

Solution of problem #18.5

$I(t) = kt$, $-\infty < t < \infty$; we want to calculate $\vec{A}$:

$$\vec{A}(t) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}(t_r)}{r}\,dl = \frac{\mu_0 k}{4\pi}\int \frac{t - r/c}{r}\,d\vec{l} = \frac{\mu_0 k}{4\pi}\left\{ t\int \frac{d\vec{l}}{r} - \frac{1}{c}\int d\vec{l} \right\}$$

For the complete loop $\int d\vec{l} = 0$, so (r is measured from the center)

$$\vec{A} = \frac{\mu_0 k t}{4\pi}\left\{ \frac{1}{a}\int_{\text{semicircle } a} d\vec{l} + \frac{1}{b}\int_{\text{semicircle}} d\vec{l} + 2\hat{x}\int_a^b \frac{dx}{x} \right\}$$

Note that for one segment $x>0$ and the other $x<0$.

$$\int_{\text{semicircle } a} d\vec{l} = 2a\hat{x} \quad , \quad \int_{\text{semicircle } b} d\vec{l} = -2b\hat{x}$$

$$\boxed{\vec{A} = \frac{\mu_0 k t}{4\pi}\left\{ 2\hat{x} - 2\hat{x} + 2\hat{x}\ln\frac{b}{a} \right\} = \frac{\mu_0 k t}{2\pi}\ln\frac{b}{a}\,\hat{x}}$$

$$\boxed{\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0 k}{2\pi}\ln\frac{b}{a}\,\hat{x}} \qquad (\text{note that } V=0)$$

The changing magnetic field induces the electric field. Since we only know $\vec{A}$ at one point (the center), we cannot compute $\vec{\nabla}\times\vec{A}$ to get $\vec{B}$.