Solution of problem # 19.1

At time t the charge is at $\vec{r}(t) = a\left[\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}\right]$.

Hence, $\vec{v}(t) = \omega a\left[-\sin(\omega t)\hat{x} + \cos(\omega t)\hat{y}\right]$ and

$$\vec{\mathscr{r}} = z\hat{z} - \vec{r}(t_r) = z\hat{z} - a\left[\cos(\omega t_r)\hat{x} + \sin(\omega t_r)\hat{y}\right].$$

Consequently, $\mathscr{r}^2 = z^2 + a^2$, and $\mathscr{r} = \sqrt{z^2+a^2}$,

$$\hat{\mathscr{r}}\cdot\vec{v} = \frac{1}{\mathscr{r}}(\vec{\mathscr{r}}\cdot\vec{v}) = \frac{1}{\mathscr{r}}\left\{-\omega a^2\left[-\sin(\omega t_r)\cos(\omega t_r) + \cos(\omega t_r)\sin(\omega t_r)\right]\right\}$$

$$= 0$$

Hence, $\left(1 - \frac{\hat{\mathscr{r}}\cdot\vec{v}}{c}\right) = 1$

Therefore $t_r = t - \frac{\sqrt{z^2+a^2}}{c}$ and

$$\boxed{V(z,t) = \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{z^2+a^2}}}, \quad \boxed{\vec{A}(z,t) = \frac{q\omega a}{4\pi\epsilon_0 c^2}\left[-\sin(\omega t_r)\hat{x} + \cos(\omega t_r)\hat{y}\right]}$$

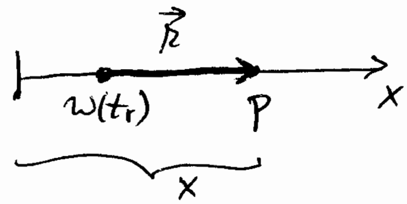
$$\frac{1}{\sqrt{z^2+a^2}}$$

Solution of problem #19.2

The fields of a moving point charge are

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[(c^2-v^2)\vec{u} + \vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right]$$

$$\vec{B}(\vec{r},t) = \frac{1}{c}\hat{\mathscr{r}}\times\vec{E}(\vec{r},t)$$



$$\vec{V} = v\hat{x} \quad , \quad \vec{a} = a\hat{x}$$

for <u>points to the right</u> of the charge

$$\hat{\mathscr{r}} = \hat{x} \quad , \quad \vec{u} = (c-v)\hat{x} \quad , \quad \vec{u}\times\vec{a} = 0 \quad , \quad \vec{\mathscr{r}}\cdot\vec{u} = \mathscr{r}(c-v)$$

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{\mathscr{r}}{\mathscr{r}^3(c-v)^3}(c^2-v^2)(c-v)\hat{x} = \frac{q}{4\pi\varepsilon_0}\frac{1}{\mathscr{r}^2}\frac{c+v}{c-v}\hat{x}$$

$$\vec{B}(\vec{r},t) = \frac{1}{c}\hat{\mathscr{r}}\times\vec{E}(\vec{r},t) = 0 .$$

For field <u>points to the left</u> of the charge,

$\hat{\mathscr{r}} = -\hat{x}$ and $\vec{u} = -(c+v)\hat{x}$, so $\vec{\mathscr{r}}\cdot\vec{u} = \mathscr{r}(c+v)$, and

$$\vec{E}(\vec{r},t) = -\frac{q}{4\pi\varepsilon_0}\frac{\mathscr{r}}{\mathscr{r}^3(c+v)^3}(c^2-v^2)(c+v)\hat{x} = -\frac{q}{4\pi\varepsilon_0}\frac{1}{\mathscr{r}^2}\left(\frac{c-v}{c+v}\right)\hat{x} ,$$

$$\vec{B}(\vec{r},t) = 0$$

Solution of problem #19.3

We know that by Gauss's law in integral form the answer has to be q/ε_0.

We know that the electric field of a point charge moving with constant velocity is

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{1-v^2/c^2}{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}\frac{\hat{R}}{R^2},$$

where $\vec{R} = \vec{r}-\vec{v}t$ and θ is the angle between $\vec{R}$ and $\vec{v}$.

$$\oint \vec{E}\cdot d\vec{a} = \frac{q(1-v^2/c^2)}{4\pi\varepsilon_0}\int \frac{R^2\sin\theta\, d\theta\, d\phi}{\left[1-\frac{v^2}{c^2}\sin^2\theta\right]^{3/2}}\frac{1}{R^2} = \frac{q(1-v^2/c^2)}{4\pi\varepsilon_0}2\pi\int_0^\pi \frac{\sin\theta\, d\theta}{\left[1-\frac{v^2}{c^2}\sin^2\theta\right]^{3/2}}$$

Note that the surface is a sphere.

Let $u=\cos\theta$, $du = -\sin\theta\, d\theta$, $\sin^2\theta = 1-u^2$

$$\oint \vec{E}\cdot d\vec{a} = \frac{q(1-v^2/c^2)}{4\pi\varepsilon_0}2\pi\int_{-1}^{1} du\frac{1}{\left[1-\frac{v^2}{c^2}+\frac{v^2}{c^2}u^2\right]^{3/2}} = \frac{q\left(1-\frac{v^2}{c^2}\right)}{2\varepsilon_0}\int_{-1}^{1}\frac{du}{\left[\frac{c^2}{v^2}-1+u^2\right]^{3/2}}\cdot\left(\frac{c}{v}\right)^3$$

Now

$$\int_{-1}^{1} du\frac{1}{\left[\frac{c^2}{v^2}-1+u^2\right]^{3/2}} = \frac{u}{\left(\frac{c^2}{v^2}-1\right)\left(\frac{c^2}{v^2}-1+u^2\right)^{1/2}}\Bigg|_{-1}^{1} = \frac{2}{\left(\frac{c^2}{v^2}-1\right)\frac{c}{v}} = \left(\frac{v}{c}\right)^3\frac{2}{1-\frac{v^2}{c^2}}$$

Hence,

$$\oint \vec{E}\cdot d\vec{a} = \frac{q\left(1-\frac{v^2}{c^2}\right)}{2\varepsilon_0}\left(\frac{c}{v}\right)^3\cdot\left(\frac{v}{c}\right)^3\frac{2}{1-\frac{v^2}{c^2}} = \boxed{\frac{q}{\varepsilon_0}}$$

✓

Solution of problem # 19.4

Consider two tiny metal spheres separated by a distance d and connected by a fine wire. The upper sphere has charge $q(t) = q_0 \cos(\omega t)$ and the lower sphere charge $-q(t)$.

(a) The dissipation is given by

$$P = I^2 R = q_0^2 \omega^2 \sin^2(\omega t) R \quad , \quad \langle P \rangle = \frac{1}{2} q_0^2 \omega^2 R$$

$\langle P \rangle$ is now equated to the radiated power

$$\langle P \rangle = \frac{\mu_0 p_0^2 \omega^4}{12 \pi c} \; .$$

Hence, $\quad R = \dfrac{\mu_0 d^2 \omega^2}{6 \pi c} \quad$, but since $\omega = \dfrac{2\pi c}{\lambda}$

$$R = \frac{\mu_0 d^2}{6\pi c}\left(\frac{2\pi c}{\lambda}\right)^2 = \frac{2}{3}\pi\left(4\pi \times 10^{-7}\right)\left(3 \cdot 10^8\right)\left(\frac{d}{\lambda}\right)^2 = 80\pi^2\left(\frac{d}{\lambda}\right)^2 \Omega$$

$$\boxed{= 789.6 \left(\frac{d}{\lambda}\right)^2 \Omega}$$

(b) For the wire with $d = 5\,cm$ and $\lambda = 1000\,m$

$$R = 790 \cdot \left(5 \times 10^{-5}\right)^2 = 2 \times 10^{-6}\ \Omega$$

Hence, R is negligible compared to the Ohmic resistance,

Solution of problem # 19.5

The potentials for an electric dipole oscillating along the z-direction are

$$V(r,\theta,t) = \frac{p_0 \cos\theta}{4\pi\varepsilon_0 r}\left\{-\frac{\omega}{c}\sin[\omega(t-r/c)] + \frac{1}{r}\cos[\omega(t-r/c)]\right\}$$

$$\vec{A}(r,\theta,t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin[\omega(t-r/c)]\,\hat{z}$$

$$\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\ , \text{ so}$$

$$\vec{\nabla}\cdot\vec{A} = -\frac{\mu_0 p_0 \omega}{4\pi}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{1}{r}\sin[\omega(t-r/c)]\cos\theta\right] + \right.$$

$$\left. + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[-\sin^2\theta\frac{1}{r}\sin[\omega(t-r/c)]\right]\right\} =$$

$$= -\frac{\mu_0 p_0 \omega}{4\pi}\left\{\frac{1}{r^2}\left[\sin[\omega(t-r/c)] - \frac{\omega r}{c}\cos[\omega(t-r/c)]\right]\cos\theta\right.$$

$$\left. - \frac{2\cancel{\sin\theta}\cos\theta}{r^2\cancel{\sin\theta}}\sin[\omega(t-r/c)]\right\} =$$

$$= \mu_0\varepsilon_0\left\{\frac{p_0\omega}{4\pi\varepsilon_0}\left(\frac{1}{r^2}\sin[\omega(t-r/c)] + \frac{\omega}{rc}\cos[\omega(t-r/c)]\right)\cos\theta\right\}$$

$$\frac{\partial V}{\partial t} = \frac{p_0\cos\theta}{4\pi\varepsilon_0 r}\left\{-\frac{\omega^2}{c}\cos[\omega(t-r/c)] - \frac{\omega}{r}\sin[\omega(t-r/c)]\right\} =$$

$$= -\frac{p_0\omega}{4\pi\varepsilon_0}\left\{\frac{1}{r^2}\sin[\omega(t-r/c)] + \frac{\omega}{rc}\cos[\omega(t-r/c)]\right\}\cos\theta$$

Hence, $\vec{\nabla}\cdot\vec{A} = -\mu_0\varepsilon_0\frac{\partial V}{\partial t}$ and the Lorentz gauge condition is satisfied