Solution of problem # 20.1

Current through the loop $\quad I(t) = I_0 \cos(\omega t)$

electrical dissipation $\quad I^2 R = I_0^2 R \cos^2(\omega t)$

$$\langle P \rangle = \frac{1}{2} I_0^2 R \qquad \text{(time average)}$$

The power radiated by the magnetic dipole is $\quad (m_0 = \pi b^2 I_0)$

$$\langle P \rangle = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3} = \frac{\mu_0 \pi^2 b^4 I_0^2 \omega^4}{12\pi c^3} \quad , \; \omega = \frac{2\pi c}{\lambda}$$

Hence,

$$\left[R = \frac{\mu_0 \pi b^4 \omega^4}{6c^3} = \frac{\mu_0 \pi b^4}{6c^3}\left(\frac{2\pi c}{\lambda}\right)^4 = \boxed{\frac{8}{3}\pi^5 \mu_0 c \left(\frac{b}{\lambda}\right)^4} \right.$$

$$R = \frac{8}{3}\pi^5 (4\pi \times 10^{-7})(3\times 10^8)\left(\frac{b}{\lambda}\right)^4 = \boxed{3.08 \cdot 10^5 \left(\frac{b}{\lambda}\right)^4 \Omega}.$$

Because $b \ll \lambda$, R goes like the fourth power of this small number. R is typically much smaller than the electric radiative resistance. For typical dimensions, $b = 5\text{cm}$ and $\lambda = 10^3 \text{m}$,

$$R = 3{,}08 \times 10^5 (5\times 10^{-5})^4 \simeq 2 \times 10^{-12}\, \Omega,$$

which is 10^6 times smaller than the electric radiative resistance (problem # 19.4).

Solution of problem #20.2

(a) For an oscillating magnetic dipole we have $V=0$ (no net charges) and

$$\vec{A} = \frac{\mu_0 m_0}{4\pi}\left(\frac{\sin\theta}{r}\right)\left\{\frac{1}{r}\cos[\omega(t-r/c)] - \frac{\omega}{c}\sin[\omega(t-r/c)]\right\}\hat{\phi}$$

Here we have used approximations 1 and 2, but not approximation 3.

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} = -\frac{\mu_0 m_0}{4\pi}\left(\frac{\sin\theta}{r}\right)\left\{\frac{1}{r}(-\omega)\sin[\omega(t-r/c)] - \frac{\omega^2}{c}\cos[\omega(t-r/c)]\right\}\hat{\phi}$$

$$\boxed{= \frac{\mu_0 m_0 \omega}{4\pi}\left(\frac{\sin\theta}{r}\right)\left\{\frac{1}{r}\sin[\omega(t-r/c)] + \frac{\omega}{c}\cos[\omega(t-r/c)]\right\}\hat{\phi}}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\phi \sin\theta\right)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\hat{\theta} =$$

$$= \frac{\mu_0 m_0}{4\pi}\left\{\frac{1}{r\sin\theta}\,\frac{2\sin\theta\cos\theta}{r}\left[\frac{1}{r}\cos[\omega(t-r/c)] - \frac{\omega}{c}\sin[\omega(t-r/c)]\right]\hat{r}\right.$$

$$\left. - \frac{\sin\theta}{r}\left[-\frac{1}{r^2}\cos[\omega(t-r/c)] + \frac{\omega}{rc}\sin[\omega(t-r/c)] + \left(\frac{\omega}{c}\right)^2\cos[\omega(t-r/c)]\right]\hat{\theta}\right\}$$

$$\boxed{\begin{aligned} = \frac{\mu_0 m_0}{4\pi}\Bigg\{ &\frac{2\cos\theta}{r^2}\left[\frac{1}{r}\cos[\omega(t-r/c)] - \frac{\omega}{c}\sin[\omega(t-r/c)]\right]\hat{r} \\ &- \frac{\sin\theta}{r}\left[-\frac{1}{r^2}\cos[\omega(t-r/c)] + \frac{\omega}{rc}\sin[\omega(t-r/c)] + \left(\frac{\omega}{c}\right)^2\cos[\omega(t-r/c)]\right]\hat{\theta}\Bigg\}\end{aligned}}$$

(b) The Poynting vector is (denoting $u = -\omega(t-r/c)$)

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{\mu_0 m_0^2\omega}{16\pi^2}\,\frac{\sin\theta}{r^2}\left\{\frac{2\cos\theta}{r}\left[\left(\frac{\omega}{c}\right)^2\cos u\sin u - \frac{\omega}{cr}\sin^2 u + \frac{\omega}{cr}\cos^2 u\right.\right.$$

$$\left. - \frac{1}{r^2}\sin u\cos u\right]\hat{\theta} + \sin\theta\left[\left(-2\left(\frac{\omega}{c}\right)^2\frac{1}{r} + \frac{1}{r^3}\right)\sin u\cos u + \left(\frac{\omega}{c}\right)^3\cos^2 u + \right.$$

$$\left.\left. + \left(\frac{\omega}{c}\right)\frac{1}{r^2}\left(\sin^2 u - \cos^2 u\right)\right]\hat{r}\right\}$$

problem #20.2 continues

$$\vec{S} = \frac{\mu_0 m_0^2 \omega^3}{16\pi^2 c^2}\left(\frac{\sin\theta}{r^2}\right)\left\{ \frac{2\cos\theta}{r}\left[\left(1-\left(\frac{c}{\omega}\right)^2\frac{1}{r^2}\right)\sin u\cos u + \frac{c}{\omega r}\left(\cos^2 u - \sin^2 u\right)\right]\hat{\theta}\right.$$

$$\left. + \sin\theta\left[\left(-\frac{2}{r} + \left(\frac{c}{\omega}\right)^2\frac{1}{r^3}\right)\sin u\cos u + \frac{\omega}{c}\cos^2 u + \frac{c}{\omega r^2}\left(\sin^2 u - \cos^2 u\right)\right]\hat{r}\right\}$$

Now we average over time to obtain the intensity;

note that $\langle \sin u \cos u\rangle = 0$, $\langle\cos^2 u\rangle = \langle \sin^2 u\rangle = \frac{1}{2}$

$$\boxed{\langle\vec{S}\rangle = \frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3}\,\frac{\sin^2\theta}{r^2}\,\hat{r}}$$

This is the same result obtained in class with approximation 3.

Solution of problem # 20.3

The power radiated is given by

$$P = \frac{dW_r}{dt} = \frac{\mu_0 q^2 a^2}{6\pi c} \qquad \text{(Larmor formula)}$$

The acceleration is due to the Coulomb force

$$ma = F = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{x^2} \quad , \quad a = \frac{k}{x^2} \quad \text{with } k = \frac{qQ}{4\pi\varepsilon_0 m} .$$

The energy radiated along the entire path is twice that radiated on the way out

$$W_r = \frac{\mu_0 q^2}{6\pi c}\int \frac{k^2}{x^4}\,dt = \frac{\mu_0 q^2 2k^2}{6\pi c}\int_{x_0}^{\infty}\frac{1}{x^4}\frac{dt}{dx}\,dx ,$$

where x_0 is the distance of closest approach. We now use the conservation of energy to determine $v = \frac{dx}{dt}$ and x_0. Here we assume that the radiated energy is only a small fraction of the total energy.

$$W_0 = E = \frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{1}{4\pi\varepsilon_0}\frac{qQ}{x} \qquad (v_0 \text{ is the velocity at "infinity"})$$

$$v^2 = v_0^2 - 2\frac{qQ}{4\pi\varepsilon_0}\frac{1}{x}\frac{1}{m} = v_0^2 - \frac{2k}{x} \quad ; \quad x_0 = \frac{2k}{v_0^2} .$$

$$W_r = \frac{\mu_0 q^2}{6\pi c}2k^2\int_{x_0}^{\infty}\frac{dx}{x^4\sqrt{v_0^2 - 2k/x}} = \frac{\mu_0 q^2}{6\pi c}\frac{2k^2}{\sqrt{2k}}\int_{x_0}^{\infty}\frac{dx}{x^4\sqrt{\frac{1}{x_0}-\frac{1}{x}}} =$$

$$= \frac{\mu_0 q^2}{6\pi c}\frac{2k^2}{\sqrt{2k}}\frac{16}{15\,x_0^{5/2}} = \frac{\mu_0 q^2 v_0^5}{45\pi c}\frac{k^2}{(2k)^3}16 = \frac{2\mu_0 q^2 v_0^5}{45\pi c\,k} =$$

$$= \frac{2\mu_0 q v_0^5}{45\pi c}\frac{4\pi\varepsilon_0 m}{Q} = \frac{8 q v_0^5 m}{45 c^3 Q}$$

$$f = \frac{W_r}{W_0} = \frac{8 q v_0^5 m}{45 c^3 Q}\frac{2}{m v_0^2} = \boxed{\frac{16 q}{45 Q}\left(\frac{v_0}{c}\right)^3}$$

Solution of problem # 20.4

(a) Coulomb force = centrifugal force : $\frac{1}{4\pi\varepsilon_0}\frac{q^2}{r^2} = m\frac{v^2}{r}$

$v = \sqrt{\frac{1}{4\pi\varepsilon_0}\frac{q^2}{mr}}$. At the beginning $r_0 = 0{,}5\,\mathring{A}$ (Bohr radius)

$$\frac{v}{c} = \left[\frac{(1.6\times10^{-19})^2}{4\pi(8.85\times10^{-12})(9.11\times10^{-31})(5\cdot10^{-11})}\right]^{1/2}\frac{1}{3\cdot10^8} = 0{,}0075 .$$

This is a nonrelativistic velocity. When the radius is one hundredth of the Bohr radius, then v/c is only 10 times larger, i.e. 0.075. Hence, for most of the trip the velocity is nonrelativistic and we can use Larmor's formula.

(b) From the Larmor formula $(a = v^2/r)$

$$P = \frac{\mu_0 q^2}{6\pi c}a^2 = \frac{\mu_0 q^2}{6\pi c}\left(\frac{v^2}{r}\right)^2 = \frac{\mu_0 q^2}{6\pi c}\left(\frac{1}{4\pi\varepsilon_0}\frac{q^2}{mr^2}\right)^2 , \quad P = -\frac{dU}{dt}$$

$$U = U_{kin} + U_{pot} = \frac{1}{2}mv^2 - \frac{1}{4\pi\varepsilon_0}\frac{q^2}{r} = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{q^2}{r} - \frac{1}{4\pi\varepsilon_0}\frac{q^2}{r} = -\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{q^2}{r}$$

This is a consequence of the virial theorem

Hence, $-\frac{dU}{dt} = -\frac{1}{8\pi\varepsilon_0}\frac{q^2}{r^2}\frac{dr}{dt} = P = \frac{\mu_0 q^2}{6\pi c}\left(\frac{1}{4\pi\varepsilon_0}\frac{q^2}{mr^2}\right)^2$ and

$$\frac{dr}{dt} = -\frac{\mu_0 q^2}{6\pi c}\left(\frac{1}{4\pi\varepsilon_0}\frac{q^2}{mr^2}\right)^2\frac{8\pi\varepsilon_0 r^2}{q^2} = -\frac{1}{3c}\left(\frac{q^2}{2\pi\varepsilon_0 mc}\right)^2\frac{1}{r^2} \quad ; \quad \text{so}$$

$$dt = -3c\left(\frac{2\pi\varepsilon_0 mc}{q^2}\right)^2 r^2 dr \quad ; \quad \boxed{t = -3c\left(\frac{2\pi\varepsilon_0 mc}{q^2}\right)^2\int_{r_0}^{0} dr\, r^2 = c\left(\frac{2\pi\varepsilon_0 mc}{q^2}\right)^2 r_0^3}$$

$$= 3\cdot10^8\left[\frac{2\pi(8.85\cdot10^{-12})(9.11\cdot10^{-31})(3\cdot10^8)}{(1.6\cdot10^{-19})^2}\right]^2(5\cdot10^{-11})^3 = \boxed{1.3\cdot10^{-11}\,\text{s}}$$

Solution of problem # 20.5

$$\vec{p} = \varepsilon_0 \int_{\mathcal{V}} (\vec{E} \times \vec{B})\, d\tau = -\varepsilon_0 \int_{\mathcal{V}} (\vec{\nabla} V) \times \vec{B}\, d\tau$$

$$\vec{\nabla} \times (V\vec{B}) = (\vec{\nabla} V) \times \vec{B} + V(\vec{\nabla} \times \vec{B})$$

$$\vec{p} = -\varepsilon_0 \int_{\mathcal{V}} \left[\vec{\nabla} \times (V\vec{B}) - V(\vec{\nabla} \times \vec{B})\right] d\tau$$

Note that $\displaystyle\int_{\mathcal{V}} (\vec{\nabla} \times \vec{v})\, d\tau = -\oint_S \vec{v} \times d\vec{a}$ for any $\vec{v}$

$$\vec{p} = \varepsilon_0 \oint_S V\vec{B} \times d\vec{a} + \varepsilon_0 \mu_0 \int_{\mathcal{V}} V\vec{J}\, d\tau$$

The surface integral vanishes for infinite space because $\vec{B} \to 0$ as $r \to \infty$;

$\varepsilon_0 \mu_0 = \frac{1}{c^2}$:

$$\vec{p} = \frac{1}{c^2} \int_{\mathcal{V}} V\vec{J}\, d\tau$$

Expand: $V(\vec{r}) = V(\vec{0}) + (\vec{\nabla} V) \cdot \vec{r} = V(\vec{0}) - \vec{E}(\vec{0}) \cdot \vec{r}$,

$$\vec{p} = \frac{1}{c^2} V(\vec{0}) \int \vec{J}\, d\tau - \frac{1}{c^2} \int [\vec{E}(\vec{0}) \cdot \vec{r}]\, \vec{J}\, d\tau$$

For a current loop, $\int \vec{J}\, d\tau \to \int \vec{I}\, dl = I \int d\vec{l} = \vec{0}$

$$\int [\vec{E}(\vec{0}) \cdot \vec{r}]\, \vec{J}\, d\tau \Rightarrow \int [\vec{E}(\vec{0}) \cdot \vec{r}]\, \vec{I}\, dl = I \int [\vec{E}(\vec{0}) \cdot \vec{r}]\, d\vec{l} = I\vec{a} \times \vec{E}(\vec{0}) = \vec{m} \times \vec{E},$$

where $\vec{m} = I\vec{a}$

So

$$\boxed{\vec{p} = -\frac{1}{c^2} (\vec{m} \times \vec{E})}$$