Solution of problem # 21.1

(a) The rocket clock runs slow; the earth clock reads $\gamma t = \frac{1}{\sqrt{1-v^2/c^2}} \cdot 1\text{hr}$

We have $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-9/25}} = \frac{5}{4}$; hence according to earth clocks the signal was sent $\boxed{1\text{hr } 15\text{min}}$ after take-off.

(b) According to the earth observer the rocket is now a distance $v\gamma t$ $\left(\frac{3}{5}c\right)\left(\frac{5}{4}\right)1\text{hr} = \frac{3}{4}c\text{ hr}$ (three-quarters of a light hour) away. The light signal will therefore take $\frac{3}{4}$hr to return to earth. Since it left 1hr 15min after departure, the light signal reaches earth $\boxed{2\text{hrs}}$ after take-off.

(c) Earth clocks run slow when observed from the rocket:

$$t_{\text{rocket}} = \gamma \cdot 2\text{hrs} = \frac{5}{4} \cdot 2\text{hrs} = \boxed{2.5\text{ hrs}}$$

Solution of problem # 21.2

$$-\bar{a}^0\bar{b}^0+\bar{a}^1\bar{b}^1+\bar{a}^2\bar{b}^2+\bar{a}^3\bar{b}^3=-\gamma^2(a^0-\beta a^1)(b^0-\beta b^1)+$$

$$+\gamma^2(a^1-\beta a^0)(b^1-\beta b^0)+a^2b^2+a^3b^3=$$

$$=-\gamma^2\left(a^0b^0+\beta^2a^1b^1-\cancel{\beta a^0b^1}-\cancel{\beta a^1b^0}-a^1b^1-\beta^2a^0b^0+\cancel{\beta a^1b^0}+\cancel{\beta a^0b^1}\right)+$$

$$+a^2b^2+a^3b^3=-\gamma^2(1-\beta^2)a^0b^0+\gamma^2(1-\beta^2)a^1b^1+a^2b^2+a^3b^3=$$

$$=-a^0b^0+a^1b^1+a^2b^2+a^3b^3 \qquad \text{QED}$$

Solution of problem #21.3

(a) $\bar{x} = x - vt$

$\bar{y} = y$

$\bar{z} = z$

$\bar{t} = t$

$$\begin{pmatrix} c\bar{t} \\ \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

(b)

$$\Lambda_x = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Lambda_y = \begin{pmatrix} \bar{\gamma} & 0 & -\bar{\gamma}\bar{\beta} & 0 \\ 0 & 1 & 0 & 0 \\ -\bar{\gamma}\bar{\beta} & 0 & \bar{\gamma} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(c)

$$\Lambda_y \Lambda_x = \begin{pmatrix} \bar{\gamma} & 0 & -\bar{\gamma}\bar{\beta} & 0 \\ 0 & 1 & 0 & 0 \\ -\bar{\gamma}\bar{\beta} & 0 & \bar{\gamma} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma\bar{\gamma} & -\gamma\bar{\gamma}\beta & -\bar{\gamma}\bar{\beta} & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ -\gamma\bar{\gamma}\bar{\beta} & \gamma\bar{\gamma}\beta\bar{\beta} & \bar{\gamma} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Lambda_x \Lambda_y = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{\gamma} & 0 & -\bar{\gamma}\bar{\beta} & 0 \\ 0 & 1 & 0 & 0 \\ -\bar{\gamma}\bar{\beta} & 0 & \bar{\gamma} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma\bar{\gamma} & -\gamma\beta & -\gamma\bar{\gamma}\bar{\beta} & 0 \\ -\gamma\bar{\gamma}\beta & \gamma & \gamma\bar{\gamma}\beta\bar{\beta} & 0 \\ -\bar{\gamma}\bar{\beta} & 0 & \bar{\gamma} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

It is the transposed result. Hence, the order of the matrices matters.

Solution of problem #21.4

(a) $\left(1-\frac{u^2}{c^2}\right)\eta^2 = u^2$; $u^2\left(1+\frac{\eta^2}{c^2}\right)=\eta^2$, $\vec{u} = \frac{1}{\sqrt{1+\eta^2/c^2}}\vec{\eta}$

(b) $\eta^\mu\eta_\mu = -(\eta^0)^2+\eta^2 = \frac{1}{(1-u^2/c^2)}\left(-c^2+u^2\right) = \boxed{-c^2}$

the four-vector is <u>timelike</u>

(c) From part (a) $u = \frac{1}{\sqrt{1+\eta^2/c^2}}\eta$; $\eta = \frac{4}{3}c$, so that

$\frac{1}{\sqrt{1+\left(\frac{\eta}{c}\right)^2}} = \frac{1}{\sqrt{1+\frac{16}{9}}} = \frac{3}{5}$, and hence $u = \frac{3}{5}(4\times10^8)\frac{m}{s} =$

$= 2.4\times10^8\,\frac{m}{s} < 2.5\times10^8\,\frac{m}{s}$

Hence, the speed is less than the speed limit, and the driver is <u>innocent</u>.

Solution of problem # 21.5

(a) Initial momentum: $E^2 - p^2c^2 = m^2c^4$; $p^2c^2 = (2mc^2)^2 - m^2c^4 =$
$= 3m^2c^4 \quad \Rightarrow \quad p = \sqrt{3}\, mc$

Initial energy: $2mc^2 + mc^2 = 3mc^2$

Both, energy and momentum are conserved; the final energy is then $3mc^2$ and the final momentum is $\sqrt{3}\, mc$

$$E^2 - p^2c^2 = (3mc^2)^2 - (\sqrt{3}\, mc)^2 c^2 = 6m^2c^4 = M^2c^4 \Rightarrow \boxed{M = \sqrt{6}\, m}$$

Since $\sqrt{6}\, m > 2m$, in this process some kinetic energy is converted into rest energy, so $M > 2m$.

$$\boxed{v = \frac{pc^2}{E} = \frac{\sqrt{3}\, mc c^2}{3mc^2} = \frac{c}{\sqrt{3}}}.$$

(b) Energy of the pion: $E^2 = p^2c^2 + m^2c^4 = \frac{9}{16} m^2c^4 + m^2c^4 = \frac{25}{16} m^2c^4$,
$E = \frac{5}{4} mc^2$

Conservation of energy: $\frac{5}{4} mc^2 = E_A + E_B$

Conservation of momentum: $\frac{3}{4} mc = p_A + p_B = \frac{E_A}{c} - \frac{E_B}{c}$ or

$\frac{3}{4} mc^2 = E_A - E_B$

solve for E_A and E_B: $\boxed{E_A = mc^2}$, $\boxed{E_B = \frac{1}{4} mc^2}$