Vector Formulas

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\nabla\times\nabla\psi = 0$$

$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$

$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$

$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

If $\mathbf{x}$ is the coordinate of a point with respect to some origin, with magnitude $r = |\mathbf{x}|$, $\mathbf{n} = \mathbf{x}/r$ is a unit radial vector, and $f(r)$ is a well-behaved function of r, then

$$\nabla\cdot\mathbf{x} = 3 \qquad \nabla\times\mathbf{x} = 0$$

$$\nabla\cdot[\mathbf{n}f(r)] = \frac{2}{r}f + \frac{\partial f}{\partial r} \qquad \nabla\times[\mathbf{n}f(r)] = 0$$

$$(\mathbf{a}\cdot\nabla)\mathbf{n}f(r) = \frac{f(r)}{r}[\mathbf{a} - \mathbf{n}(\mathbf{a}\cdot\mathbf{n})] + \mathbf{n}(\mathbf{a}\cdot\mathbf{n})\frac{\partial f}{\partial r}$$

$$\nabla(\mathbf{x}\cdot\mathbf{a}) = \mathbf{a} + \mathbf{x}(\nabla\cdot\mathbf{a}) + i(\mathbf{L}\times\mathbf{a})$$

where $\mathbf{L} = \frac{1}{i}(\mathbf{x}\times\nabla)$ is the angular-momentum operator.

Theorems from Vector Calculus

In the following ϕ, ψ, and $\mathbf{A}$ are well-behaved scalar or vector functions, V is a three-dimensional volume with volume element d^3x, S is a closed two-dimensional surface bounding V, with area element da and unit outward normal $\mathbf{n}$ at da.

$$\int_V \nabla \cdot \mathbf{A}\, d^3x = \int_S \mathbf{A} \cdot \mathbf{n}\, da \qquad \text{(Divergence theorem)}$$

$$\int_V \nabla\psi\, d^3x = \int_S \psi\mathbf{n}\, da$$

$$\int_V \nabla \times \mathbf{A}\, d^3x = \int_S \mathbf{n} \times \mathbf{A}\, da$$

$$\int_V (\phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi)\, d^3x = \int_S \phi\mathbf{n} \cdot \nabla\psi\, da \qquad \text{(Green's first identity)}$$

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, d^3x = \int_S (\phi\nabla\psi - \psi\nabla\phi) \cdot \mathbf{n}\, da \qquad \text{(Green's theorem)}$$

In the following S is an open surface and C is the contour bounding it, with line element $d\mathbf{l}$. The normal $\mathbf{n}$ to S is defined by the right-hand-screw rule in relation to the sense of the line integral around C.

$$\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, da = \oint_C \mathbf{A} \cdot d\mathbf{l} \qquad \text{(Stokes's theorem)}$$

$$\int_S \mathbf{n} \times \nabla\psi\, da = \oint_C \psi\, d\mathbf{l}$$