NAME:

ELECTRICITY AND MAGNETISM II
First Midterm Exam (Spring 2016)

Show all your work to receive full credit

# 1: (15 points)
Two coils are wrapped around a cylindrical form in such a way that the same flux passes through every turn of both coils. The primary coil has has $N_1$ turns and the secondary has $N_2$ turns (see figure). This ideal transformer takes an input AC voltage of amplitude $V_1$, and delivers an output voltage of amplitude $V_2$, which is determined by the turns ratio $N_2/N_1$.

(a) In an ideal transformer, the same flux passes through all turns of the primary and of the secondary. Show that in this case $M^2 = L_1 L_2$, where $M$ is the mutual inductance of the coils, and $L_1$, $L_2$ are their individual self-inductances. (3 pts.)

(b) Suppose the primary is driven with AC voltage $V_{in} = V_1 \cos(\omega t)$, and the secondary is connected to a resistor, $R$. Show that the two currents satisfy the relations

$$L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt} = V_1 \cos(\omega t) ; \quad L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt} = -I_2 R .$$ (3 pts.)
(c) Using the result in (b), solve these equations for $I_1(t)$ and $I_2(t)$. (Assume $I_1$ has no DC component.) (3 pts.)

(d) Show that the output voltage ($V_{out} = I_2R$) divided by the input voltage ($V_{in}$ is equal to the turns ratio: $V_{out}/V_{in} = N_2/N_1$. (3 pts.)

(e) Calculate the input power ($P_{in} = V_{in}I_1$) and the output power ($P_{out} = V_{out}I_2$), and show that their averages over a full cycle are equal. (3 pts.)

# 2 : (10 points)

An infinitely long cylindrical tube, of radius $a$, moves at constant speed $v$ along its axis. It carries a net charge per unit length $\lambda$, uniformly distributed over its surface. Surrounding it, at radius $b$, is another cylinder, moving with the same velocity but carrying the opposite charge ($-\lambda$).

(a) Find the electric and magnetic fields between the cylinders. (2 pts.)

(b) Find the energy per unit length stored in the fields. (3 pts.)

(c) Find the momentum per unit length in the fields. (3 pts.)

(d) Find the energy per unit time transported by the fields across a plane perpendicular to the cylinders. (2 pts.)

*Hint: $g = \varepsilon_0(E \times B)$*
Solution of problem #1

(a) \( \Phi_1 = I_1 L_1 + MI_2 = N_1 \Phi \), \( \Phi_2 = I_2 L_2 + MI_1 = N_2 \Phi \)

\( \Phi = I_1 \frac{L_1}{N_1} + I_2 \frac{M}{N_1} = I_2 \frac{L_2}{N_2} + I_1 \frac{M}{N_2} \)

if \( I_1 = 0 \) \( MN_2 = L_2 N_1 \) and if \( I_2 = 0 \) \( L_1 N_2 = MN_1 \)

or \( \frac{M}{L_1} = \frac{N_2}{N_1} = \frac{L_2}{M} \Rightarrow M^2 = L_1 L_2 \)

(b) \( -E_1 = \frac{d}{dt} \Phi_1 = L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt} = V_1 \cos(\omega t) \)

\( -E_2 = \frac{d}{dt} \Phi_2 = L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt} = -I_2 R \)

(c) multiply first eqn. by \( L_2 \)

\( L_1 L_2 \frac{dI_1}{dt} + L_2 M \frac{dI_2}{dt} = L_2 V_1 \cos(\omega t) \)

and insert \( L_2 \frac{dI_2}{dt} \) from second eqn.

\( L_1 L_2 \frac{dI_1}{dt} + M (-I_2 R - M \frac{dI_1}{dt}) = L_2 V_1 \cos(\omega t) \),

since \( L_1 L_2 = M^2 \). Hence

\[ I_2(t) = -\frac{L_2 V_1}{M} \cos(\omega t) \]

Now insert \( I_2(t) \) into the first eqn.:

\( L_1 \frac{dI_1}{dt} + M \frac{L_2 V_1}{M} \cos(\omega t) = V_1 \cos(\omega t) \)

\( \frac{dI_1}{dt} = -\frac{L_2 V_1}{L_1 R} \omega \sin(\omega t) + \frac{V_1}{L_1} \cos(\omega t) \)

\[ I_3(t) = \frac{L_2 V_1}{L_1 R} \cos(\omega t) + \frac{V_1}{L_1} \frac{1}{\omega} \sin(\omega t) = \frac{V_1}{L_1} \left[ \frac{1}{\omega} \sin(\omega t) + \frac{L_2}{R} \cos(\omega t) \right] \]
problem # 1 continues

(d) \[ \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{I_2 R}{V_i \cos(\omega t)} = - \frac{L_2 V_i \cos(\omega t)}{M \cos(\omega t)} = - \frac{L_2}{M} = - \frac{N_2}{N_1} \quad \text{(from (a))} \]

(e) \[ P_{\text{in}} = V_{\text{in}} I_1 = V_i \cos(\omega t) \frac{V_i}{L_1} \left[ \frac{1}{\omega} \sin(\omega t) + \frac{L_2}{R} \cos(\omega t) \right] = \]
\[ = \frac{V_i^2}{L_1} \left[ \frac{1}{\omega} \sin(\omega t) \cos(\omega t) + \frac{L_2}{R} \cos^2(\omega t) \right] \]

\text{time average} \quad \langle P_{\text{in}} \rangle = \frac{V_i^2 L_2}{2 L_1 R} \]

\[ P_{\text{out}} = V_{\text{out}} I_2 = I_2^2 R = \frac{L_2 V_i^2}{M^2 R} \cos^2(\omega t) \]

\text{time average} \quad \langle P_{\text{out}} \rangle = \frac{L_2 V_i^2}{2 M^2 R} = \frac{1}{2} \frac{V_i^2}{R} \frac{L_2}{M^2} = \frac{1}{2} \frac{V_i^2 L_2}{L_1 L_2} = \]
\[ = \frac{1}{2} \frac{V_i^2}{R} \frac{L_2}{L_1} = \langle P_{\text{in}} \rangle \quad \checkmark \]
Solution of problem #2

(a) The fields are zero for \( s < a \) and \( s > b \). Between the cylinders,
\[
\vec{E} = \frac{1}{2\pi \varepsilon_0} \frac{\lambda}{s} \hat{\mathbf{i}}
\]
cylinder of radius \( s \) as Gaussian surface
\[
2\pi s \int E = \lambda l / \varepsilon_0
\]
\[
\vec{B} = \frac{\mu_0 I}{2\pi} \hat{\phi}
\]
Amperian loop of radius \( s \)
\[
2\pi s B = \mu_0 I = \mu_0 \lambda l
\]

(b) \[
U = \frac{1}{2} \left[ \varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right] = \frac{1}{2} \left[ \varepsilon_0 \left( \frac{\lambda}{2\pi \varepsilon_0 s} \right)^2 + \frac{1}{\mu_0} \left( \frac{\mu_0 I}{2\pi} \frac{1}{s} \right)^2 \right] = \frac{\lambda^2}{8\pi^2 \varepsilon_0} \left( 1 + \varepsilon_0 \mu_0 \frac{V^2}{c^2} \right) \frac{1}{s^2}
\]
Integrating over the volume between the cylinders
\[
W = \frac{\lambda^2}{8\pi^2 \varepsilon_0} \left( 1 + \varepsilon_0 \frac{V^2}{c^2} \right) \int_a^b \frac{1}{s^2} (2\pi s) ds = \frac{\lambda^2}{4\pi \varepsilon_0} \left( 1 + \frac{V^2}{c^2} \right) \ln \left( \frac{b}{a} \right)
\]

(c) \[
\vec{\mathbf{j}} = \varepsilon_0 (\vec{E} \times \vec{B}) = \varepsilon_0 \left( \frac{\lambda}{2\pi \varepsilon_0 s} \right) \left( \frac{\mu_0 I}{2\pi} \frac{1}{s} \right) \hat{z} = \frac{\mu_0}{4\pi^2} \frac{V^2}{s^2} \hat{z}
\]
\[
\vec{P} = \frac{\mu_0 V^2}{4\pi^2} \frac{1}{s} \int_a^b 2\pi s ds = \frac{\mu_0 V^2}{2\pi} \ln \left( \frac{b}{a} \right) \hat{z}
\]

(d) \[
\vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B}) = \frac{1}{\mu_0 \varepsilon_0} \vec{\mathbf{j}} = c \vec{\mathbf{j}}
\]
\[
\frac{dW}{dt} = \int \vec{S} \cdot d\vec{S} = \frac{\mu_0 V^2}{4\pi^2 \varepsilon_0} \int_a^b \frac{ds}{s^2} (2\pi s) = \frac{\lambda^2 V^2}{2\pi \varepsilon_0} \ln \left( \frac{b}{a} \right)
\]