Solution of problem #12.1

(a) \( V = \frac{Q}{C} = IR \); positive \( I \) means the charge on the capacitor decreases; hence \( \frac{dQ}{dt} = -I = -\frac{Q}{RC} \). Hence, \( Q(t) = Q_0 e^{-t/RC} \), but \( Q_0 = Q(0) = CV_0 \), so

\[ Q(t) = CV_0 e^{-t/RC} \]

It follows that

\[ I(t) = -\frac{dQ}{dt} = CV_0 \frac{1}{RC} e^{-t/RC} = \frac{V_0}{R} e^{-t/RC} \]

(b) Initial energy stored in the capacitor: \( W = \frac{1}{2} CV_0^2 \)

The energy delivered to the resistor is

\[ \int_0^\infty dt \, P = \int_0^\infty dt \, I^2 R = \frac{V_0^2}{R} \int_0^\infty dt \, e^{-2t/RC} = \frac{V_0^2}{R} \left[ -\frac{RC}{2} e^{-2t/RC} \right]_0^\infty = \frac{V_0^2 C}{2} \checkmark \]

equal to the initial energy stored in the capacitor.

(c) \( V_0 = IR + \frac{Q}{C} \); now positive \( I \) means \( Q \) is increasing.

\[ \frac{dQ}{dt} = \frac{V_0}{R} - \frac{Q}{RC} = \frac{1}{RC} (V_0 C - Q) \]

integrate

\[ \ln(Q - CV_0) = -\frac{1}{RC} t + \text{const} \quad ; \quad Q(t) = CV_0 + k e^{-t/RC} \]

but \( Q(0) = 0 \) \( \Rightarrow \) \( k = -CV_0 \) and

\[ Q(t) = CV_0 \left( 1 - e^{-t/RC} \right) \]

\[ I(t) = \frac{dQ}{dt} = CV_0 \frac{1}{RC} e^{-t/RC} = \frac{V_0}{R} e^{-t/RC} \], as in (a).
(d) Energy from the battery: \[ \int_{0}^{\infty} V_0 I \, dt = \frac{V_0^2}{R} \int_{0}^{\infty} e^{-t/RC} \, dt = \frac{V_0^2}{R} \left[ -e^{-t/RC} \right]_{0}^{\infty} = \frac{V_0^2}{RC} \]

Since the current in (c) is the same as in (a), also the energy delivered to the resistor is the same \[ \frac{1}{2} CV_0^2 \]

The final energy in the capacitor is again \[ \frac{1}{2} CV_0^2 \].

Half the energy from the battery goes to the capacitor and the other half to the resistor.
Solution of problem # 12.2

(a) \( I = \int \mathbf{J} \cdot d\mathbf{a} \), where the integral is taken over a surface enclosing the positively charged conductor, e.g. "s". By \( \mathbf{J} = \sigma \mathbf{E} \), and with Gauss's law \( \int \mathbf{E} \cdot d\mathbf{a} = \frac{Q}{\varepsilon_0} \), we have

\[ I = \sigma \int \mathbf{E} \cdot d\mathbf{a} = \frac{Q}{\varepsilon_0} \]

Now \( Q = CV \) and \( V = IR \), so that \( I = \frac{Q}{\varepsilon_0} CRI \) or \( R = \frac{\varepsilon_0}{\sigma C} \)

(b) \( Q = CV = CRI \) \Rightarrow \frac{dQ}{dt} = -I = -\frac{1}{CR} Q \),

\( Q(t) = Q_0 e^{-t/RC} \) or since \( V = \frac{Q}{C} \), \( V(t) = V_0 e^{-t/RC} \).

The time constant is \( \tau = RC = \frac{\varepsilon_0}{\sigma} \) (from (a)).
Solution of problem # 12.3

(a) \( E = - \frac{d\Phi}{dt} = - Bl \frac{dx}{dt} = - Blv \)

\( x \) is the distance between resistor and bar

\( E = RI \quad \Rightarrow \quad I = \frac{Blv}{R} \)

\( \vec{N} \times \vec{B} \) is upward, so the current is downward through the resistor

(b) \( \begin{bmatrix} F = I l B = \frac{B^2 l^2 v}{R} \end{bmatrix} \) to the left (opposite to \( \vec{v} \))

(c) \( F = ma = m \frac{dv}{dt} = - \frac{B^2 l^2 v}{R} \quad \Rightarrow \quad \frac{dv}{dt} = - \frac{B^2 l^2}{mR} v \)

\( v = v_0 \exp \left[ - \frac{B^2 l^2}{mR} t \right] \)

(d) Initial KE of bar is \( \frac{1}{2} m v_0^2 \). The energy goes into the resistor.

The power delivered to the resistor is \( I^2 R \), so

\( \frac{dW}{dt} = I^2 R = \frac{B^2 l^2 v^2}{R} = \frac{B^2 l^2}{R} v_0^2 \exp \left[ -2 \frac{B^2 l^2}{mR} t \right] \)

The total energy delivered to the resistor is

\( W = \frac{B^2 l^2}{R} v_0^2 \int_0^\infty dt \exp \left[ -2 \frac{B^2 l^2}{mR} t \right] = \frac{m v_0^2}{2} \left[ - \exp \left[ -2 \frac{B^2 l^2}{mR} t \right] \right]_0^\infty = \frac{m v_0^2}{2} \)
Solution of problem #12.4

(a) The field of the long wire is
\[ \vec{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi} \]
\[ \Phi = \int \vec{B} \cdot d\vec{a} = \frac{\mu_0 I}{2\pi} \int \frac{s+a}{s} \left( \alpha ds \right) = \frac{\mu_0 I}{2\pi} a \ln \frac{s+a}{s} \]

(b) \[ \mathcal{E} = -\frac{d\Phi}{dt} = -\frac{\mu_0 I a}{2\pi} \frac{d}{dt} \ln \frac{s+a}{s} \quad \frac{ds}{dt} = v \]

hence
\[ \mathcal{E} = -\frac{\mu_0 I a}{2\pi} \left[ \frac{1}{s+a} - \frac{1}{s} \right] v = \frac{\mu_0 I a^2 v}{2\pi s(s+a)} \]

The field points out of the page, so the force on a charge in the nearby side of the square is to the right (\( F = F' x B \)). In the far side it is also to the right, however the field is weaker. Hence, the current flows counterclockwise.

(c) If the loop is pulled to the right at speed \( v \), the flux does not change with time. If the flux is constant: \[ \mathcal{E} = 0 \]
Solution of problem # 12.5

\[ \vec{B}(t) = B_0 \cos(\omega t) \hat{z} \]

\[ \Phi = \pi \left( \frac{R}{2} \right)^2 B = \frac{\pi R^2}{4} B_0 \cos(\omega t) \]

\[ E = -\frac{d\Phi}{dt} = \frac{\pi R^2}{4} B_0 \omega \sin(\omega t) \]

\[ I(t) = \frac{E}{R} = \frac{\pi R^2}{4 R} B_0 \sin(\omega t) \]