Solution of problem #14.1

(a) $B = \mu_0 n I \Rightarrow \Phi = \mu_0 n I \pi R^2$ (flux through a single turn)
In a length $l$ there are $n$ turns. The total flux is
\[ \Phi = \mu_0 n^2 \pi R^2 l = LI, \]
\[ L = \mu_0 n^2 \pi R^2 \] self-inductance per unit length.

(b) $W = \frac{1}{2} LI^2 = \frac{1}{2} \mu_0 n^2 \pi R^2 I^2$

(c) $W = \frac{1}{2} \int (\vec{A} \cdot \vec{B}) dl$, $\vec{A} = (\mu_0 n I/2) s \hat{\phi}$ for $s \leq R$ from example
\[ \vec{B} = \nabla \times \vec{A} = -\frac{\partial A_\phi}{\partial z} \hat{z} + \frac{1}{R} \hat{\phi} \nabla s = \mu_0 n I \hat{\phi} \]
For one turn: $W_1 = \frac{1}{2} \frac{\mu_0 n I R}{2} (I \Delta t R)$
For $n$ turns: $W = \frac{\mu_0 \pi n^2 l R^2}{2} I^2$

(d) $W = \frac{1}{2} \mu_0 \int B^2 ds$ field is zero outside solenoid
\[ = \frac{1}{2} \mu_0 (\mu_0 n I)^2 \int_{\text{cylinder of length}} ds = \frac{1}{2} \mu_0 (\mu_0 n I)^2 \pi R^2 = \]
\[ = \mu_0 n^2 \pi R^2 \frac{l}{2} I^2 \]
Solution of problem # 14.2

(a) The magnetic field of a solenoid is
\[ \mathbf{B} = \begin{cases} \mu_0 K \mathbf{z} = \mu_0 \sigma \omega R \mathbf{z} , & s < R \\ 0 , & s > R \end{cases} \quad (K = \sigma \omega = \sigma R) \]

From symmetry \( \mathbf{E} \parallel \mathbf{\phi} \);
\[ \oint \mathbf{E} \cdot d\mathbf{l} = E(2\pi s) = -\frac{d\mathbf{\phi}}{dt} = -\frac{d}{dt}(\pi s^2 \mathbf{B}) = \begin{cases} -\pi s^2 \frac{d\mathbf{B}}{dt} & \text{if } s < R \\ -\pi R^2 \frac{d\mathbf{B}}{dt} & \text{if } s > R \end{cases} \]

\[ \mathbf{E} = -\frac{s}{2} \frac{d\mathbf{B}}{dt} \hat{\phi} \quad \text{if } s < R \quad \text{and} \quad \mathbf{E} = -\frac{R^2}{25} \frac{d\mathbf{B}}{dt} \hat{\phi} \quad \text{if } s > R \]

At the surface (s = R) we have \( \mathbf{E} = -\frac{1}{2} \mu_0 R^2 \sigma \omega \hat{\phi} \).

The torque on a length l of the cylinder is
\[ \mathbf{N} = \mathbf{r} \times \mathbf{F} = -R \left( 2\pi R l \right) \left( \frac{1}{2} \mu_0 R^2 \sigma \omega \hat{\phi} \right) = -\pi \mu_0 \sigma^2 R^4 \omega l \hat{z} \]

The work done by the field per unit length is \( W = \int \mathbf{N} \cdot d\mathbf{l} \)

\[ \frac{W}{l} = -\pi \mu_0 \sigma^2 R^4 \int (\mathbf{r} \times \mathbf{F}) (d\phi \mathbf{z}) = -\pi \mu_0 \sigma^2 R^4 \int_0^{\omega} \frac{d\phi}{dt} (d\phi \mathbf{z}) = \]

\[ = -\pi \mu_0 \sigma^2 \frac{R^4}{2} \frac{d\omega^2}{dt} = -\frac{\mu_0 \pi}{2} (\sigma R^2 \omega_l)^2 \]

This is the work done by the electric field. The work you must exert is

\[ \frac{W}{l} = \frac{\mu_0 \pi}{2} (\sigma R^2 \omega_l)^2 \quad \text{(without the minus sign)} \]
(b) \[ W = \frac{1}{2\mu_0} \int \overrightarrow{B} \cdot d\overrightarrow{\ell} = \frac{1}{2} \mu_0 k^2 \pi R^2 l \]

because the field is uniform inside the solenoid and zero outside, \( k = \sigma l + R \), so

\[ \frac{W}{q} = \frac{\pi}{2} \mu_0 (\sigma l + R)^2 \]
Solution of problem #14.3

(a) The displacement current is \( \vec{J}_d = \varepsilon \frac{\partial \vec{E}}{\partial t} \)

Parallel plate capacitor
\[ \vec{E} = \frac{\sigma}{\varepsilon} \vec{A} = \frac{1}{\varepsilon} \frac{\partial}{\partial t} \vec{A} \]

\[ \vec{J}_d = \frac{1}{A} \frac{dQ}{dt} = \frac{I}{A} \vec{A} = \frac{I}{\pi a^2} \vec{A} \]

Amperian loop of radius \( s \) (between plates): \( \hat{B} \propto \phi \)
\[ \oint \vec{B} \cdot d\vec{L} = B 2\pi s = \mu_0 I_{\text{enc}} = \mu_0 \frac{I}{\pi a^2} \pi s^2 = \mu_0 I \left( \frac{s}{a} \right)^2 \]

\[ B = \frac{\mu_0 I s^2}{2\pi a^2}, \quad \hat{B} = \frac{\mu_0 I s}{2\pi a^2} \hat{\phi} \]

(b) (i) \[ \vec{E} = \frac{\sigma(t)}{\varepsilon} \vec{A}, \quad \sigma(t) = \frac{Q(t)}{\pi a^2} = \frac{I t}{\pi a^2} \quad (I = \text{const}) \]
\[ \vec{E} = \frac{I t}{\pi a^2 \varepsilon} \vec{A} \]

(ii) \[ I_{\text{enc}} = \oint \vec{E} \cdot d\vec{L} = \varepsilon_0 \frac{\partial E}{\partial t} \pi s^2 = \varepsilon_0 \frac{1}{\pi a^2} \frac{I}{\varepsilon_0} \frac{s^2}{2} \quad \Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi a^2} s \hat{\phi} \]

\[ \oint \vec{B} \cdot d\vec{L} = \mu_0 I_{\text{enc}} = B 2\pi s \]

\[ B = \frac{\mu_0 I}{2\pi a^2} s \hat{\phi} \]
Solution of problem #14.4

(a) In the quasistatic approximation the magnetic field is "circumferential". Hence, the electric field is longitudinal (orthogonal to the magnetic field and by symmetry).

(b) Use the amperian loop shown in the figure. Outside the coaxial tube, $B = 0$ and $E = 0$.

Hence, $\oint E \cdot dl = \mathcal{E}L = -\frac{d\Phi}{dt} = -\frac{d}{dt} \int B \cdot dl = -\frac{d}{dt} \int_S \frac{\mu_0 I}{2\pi s} l ds$,

So $\mathcal{E} = -\frac{\mu_0}{2\pi} \frac{dI}{dt} \ln \left( \frac{a}{s} \right)$

$I = I_0 \cos(\omega t), \quad \frac{dI}{dt} = -\omega I_0 \sin(\omega t)$

(c) $\mathcal{J}_d = \varepsilon_0 \frac{d\mathcal{E}}{dt} = \varepsilon_0 \frac{\mu_0 I \omega^2}{2\pi} \cos(\omega t) \ln \left( \frac{a}{s} \right)$

(d) The displacement current:

$I_d = \oint \mathcal{J}_d \cdot dl = \frac{\mu_0 \varepsilon_0}{2\pi} \omega I_0 \int_0^a \ln \left( \frac{a}{s} \right) (2\pi s) ds = \frac{\mu_0 \varepsilon_0}{2\pi} \omega I_0 \int_0^a (s \ln a - s \ln s) ds = \frac{\mu_0 \varepsilon_0 \omega I_0 a^2}{4}$

(e) $\frac{I_d}{I} = \frac{\frac{\mu_0 \varepsilon_0 \omega I_0 a^2}{4}}{I} = \left( \frac{\omega a}{2c} \right)^2$, \(a = 10^{-3}\) m, \(\frac{I_d}{I} = 0.01\) we have

\(\frac{\omega a}{2c} = \frac{1}{10} \quad \omega = \frac{2c}{10a} = \frac{3 \times 10^8 m}{5 \times 10^{-3} m} = 0.6 \times 10^{-1} \frac{m}{s} = 6 \times 10^10 \frac{m}{s} \)
Problem # 14.4 continues

\[ \gamma = \frac{\omega}{2\pi} \approx 10^{10} \, \text{Hz} \quad \text{or} \quad 100 \, \text{GHz} \]

This is a microwave frequency, way above radio frequencies.
Solution of problem #14.5

(a) Assume that magnetic monopoles exist. Gauss's and Ampère's laws are not expected to change.

Obviously, the divergence of $\mathbf{B}$ is no longer zero, but
\[ \nabla \cdot \mathbf{B} = \kappa \mathbf{P}_m \]
where $\mathbf{P}_m$ is the density of magnetic charge, and $\kappa$ is a constant (similar to $\varepsilon$ and $\mu$).

The curl of $\mathbf{E}$ becomes
\[ \nabla \times \mathbf{E} = \beta \mathbf{J}_m - \frac{\partial \mathbf{B}}{\partial t} \]
where $\mathbf{J}_m$ is the magnetic current density (flow of magnetic charge), and $\beta$ is another constant.

Assuming that the magnetic charge is conserved, then $\mathbf{P}_m$ and $\mathbf{J}_m$ satisfy the continuity equation:
\[ \nabla \cdot \mathbf{J}_m + \frac{\partial \mathbf{P}_m}{\partial t} = 0 \]
Here $\beta = -\kappa$.

We may choose $\kappa = \mu$.

The force law ( Lorentz force for magnetic charges), should be something of the form:
\[ \mathbf{F} = q_m [\mathbf{E} + \gamma_0 (\mathbf{v} \times \mathbf{E})] \]
where $\gamma_0$ needs to have the dimension of an inverse velocity squared ($v$ has the same dimension as $\mathbf{v} \mathbf{B}$). A natural candidate for $\gamma_0$ is $\frac{1}{c^2} = \mu_0 \varepsilon_0$, so that
\[ \mathbf{F} = q_e [\mathbf{E} + (\mathbf{v} \times \mathbf{B})] + q_m [\mathbf{B} - \frac{1}{c^2} (\mathbf{v} \times \mathbf{E})] \]

(the minus sign comes in because Faraday's law has $-\frac{\partial \mathbf{B}}{\partial t}$, while Maxwell's correction to Ampère's law is $+\mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$).
problem #14.5 continues

Now the magnetic analog to Coulomb’s law reads

$$\mathbf{F} = \frac{\mu_0}{4\pi} \frac{q_m q_m}{r^2} \mathbf{r}$$

To determine $\mu_0$ we first introduce an arbitrary unit of magnetic charge and then measure the force between unit charges at a given separation.

(b) From $\nabla \cdot \mathbf{B} = \mu_0 \mathbf{J}_m$ it follows that the field of a point monopole is $\mathbf{B} = \frac{\mu_0}{4\pi} \frac{q_m}{r^2} \mathbf{r}$. The force has the form $\mathbf{F} \propto q_m (\mathbf{B} - \frac{1}{c^2} \mathbf{v} \times \mathbf{E})$. The proportionality constant must be $1$ to reproduce “Coulomb’s law” for magnetic point charges at rest. Hence,

$$\mathbf{F} = q_m (\mathbf{B} - \frac{1}{c^2} \mathbf{v} \times \mathbf{E})$$