Physics 5524
Statistical Mechanics
Problem Set 3
Due: Monday, February 9 (in class)

3.1: Consider a gas of photons with dispersion $\omega(\mathbf{k}) = c|\mathbf{k}|$ where c is the speed of light. The photons are confined to a three-dimensional volume V and are in thermal equilibrium at temperature T.

(a) For this photon gas, obtain an expression for (the grand potential) $\Sigma = k_B T \ln Z$ where Z is the (grand) partition function. This expression will involve an integral over frequency which you should evaluate using contour integration discussed in the class.

(b) Show that the dependence of Σ on volume V and temperature T is of the form $\Sigma \approx VT^\alpha$ and determine the exponent α.

(c) Show that the dependence of the entropy of this gas S on volume V and temperature T is of the form $S \approx VT^\beta$, and determine the exponent β.

(d) Now assume this gas undergoes an adiabatic (i.e. $\Delta S = 0$) expansion with initial temperature $T_i = 3000K$ and final temperature $T_f = 3K$. By what factor does the volume of the gas increase? (Your result gives a good estimate for the factor by which the volume of the universe has increased since the photons which form the cosmic microwave background, which today has a temperature of $T \approx 3K$, decoupled from matter roughly 380,000 years after the big bang, when the temperature was $T \approx 3000K$.)

3.2: Consider a 2-dimensional Debye solid of area A consisting of N ions. The total number of normal modes will be $2N$. For a Debye solid the dispersion of these modes is taken to be $\omega_s(\mathbf{k}) = c|\mathbf{k}|$.

(a) Obtain an expression for the Debye wave vector k_D of this solid for a given ion density N/A. Also obtain expressions for the Debye frequency ω_D and Debye temperature Θ_D for this solid.

(b) Obtain an expression for $C_V(T)$ for this solid.

(c) Show in the low temperature limit ($T \ll \Theta_D$) that $C_V(T) \approx T^2$ and determine the prefactor.

(d) Show in the high temperature limit ($T \gg \Theta_D$) that $C_V(T) \approx 2Nk_B$.

Useful integral to consider, involving Riemann zeta function $\zeta(x)$, is $\int_0^\infty dx x^3 e^x/(e^x-1)^2 = 6\zeta(3) \approx 7.212$.