Physics 5524
Statistical Mechanics
Problem Set 6
Due: Monday, March 16 (in class)

6.1:
In 1995 Bose-Einstein condensation was realized experimentally for dilute gases of alkali atoms (Li, Na, K, Rb, Cs). Unlike ^{4}He, these gases are sufficiently dilute to be well described by the ideal Bose gas model.

In these experiments, atoms are trapped using inhomogeneous magnetic fields which couple to the atoms through the Zeeman coupling to the unpaired s-electron in the outermost shell. The resulting magnetic trap can be modeled as a three-dimensional harmonic potential $V(\mathbf{r}) = \frac{1}{2}m\omega_0^2\mathbf{r}^2$. In what follows it is convenient to subtract off the zero-point energy so that the ground state has energy $E = 0$ rather than $\frac{3}{2}\hbar\omega_0$. The energy levels are then $E_{n_x,n_y,n_z} = \hbar\omega_0(n_x + n_y + n_z)$. Use the expression which we have derived for the partition function of ideal Bose gas of N boson.

(a) Show that if $k_BT \gg \hbar\omega_0$, the sum over n_x, n_y, n_z can be approximated by an integral, provided one has properly taken into account the possibility of a macroscopic occupation, N_0, of the one-particle ground state.

(b) Using the result of (a), obtain an expression for the Bose-Einstein condensation temperature T_c for this gas. Show that if $N \gg 1$ the assumption that $k_BT_c \gg \hbar\omega_0$ is justified.

(c) Determine the temperature dependence of the condensate fraction, N_0/N, when $T \leq T_c$ for this gas.

(d) For $T < T_c$ obtain an expression for the energy of this gas as a function of N, ω_0 and T. From this determine the temperature dependence of the specific heat for $T < T_c$.

(e) For a trap with angular frequency $\omega_0 = 2\pi \times 100s^{-1}$ (a typical number) and 10^6 atoms calculate T_c in Kelvin.

These integrals may be useful:

$$\int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \frac{1}{e^{x+y+z} - 1} = \zeta(3); \quad \int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \frac{x+y+z}{e^{x+y+z} - 1} = 3\zeta(4),$$

where $\zeta(n)$ is the Riemann zeta function.

6.2: Consider a degenerate, ultra-relativistic ($mc^2 \ll cp_F$) gas of non-interacting electrons, where p_F is the Fermi momentum. In this limit, the energy of an electron is related to its momentum by $E(\mathbf{p}) = c|\mathbf{p}|$. Consider N such electrons in a volume V .

(a) At zero temperature, find the chemical potential μ and the Fermi momentum p_F for this gas as a function of N and V.
(b) Find the total internal energy of the system U at zero temperature as a function of N and V.
(c) Use the fact that at $T = 0$ the pressure is given $P = -\partial U/\partial V$ to obtain an expression for P and show that $PV = \frac{1}{3}U$ when $T = 0$.
(d) Use the partition function for this gas at finite temperature to show that $PV = \frac{1}{3}U$ for any temperature T (not just $T = 0$).